PROCEEDINGS
OF THE
NATIONAL ACADEMY OF SCIENCES
Volume 20 March 15, 1934 Number 3

SURVIVAL OF YOUNG RABBIT EMBRYOS ON ARTIFICIAL MEDIA

By A. J. Waterman[1]

Department of Biology, Brooklyn College

Communicated February 2, 1934

Explantation of young rabbit embryos of the late primitive streak and two somite stages has been made to the surface of coagulated bacteriological media to determine if a suitable artificial medium could be found which would promote growth and differentiation, and to study the effects of the strange environment upon the explants.

Of the various types of media employed thus far, two have yielded certain results: (1) nutrient gelatin consisting of gelatin 10%, peptone 0.5%, NaCl 0.85% and meat extract 0.3%; and (2) Loeffler's coagulated blood-serum containing lean veal infusion, beef blood-serum, peptone and 0.3% NaCl together with 1% uncoagulated acidic fluid. The second medium was tried with and without the acidic fluid and also with a change of pH to values of 6.2, 7 and 7.8. The technique is essentially similar to that described previously for the culture of rabbit embryos upon the surface of a medium composed of coagulated chicken plasma plus a saline extract of a nine-day chick embryo.[2,3] The prepared medium was poured either into a sterilized watch glass enclosed in a Petri dish with a ring of moistened cotton to prevent evaporation or into the bottom of the sterilized Petri dish, after which the dishes were kept in the incubator until desired. The first method was found more suitable since there was less surface to cover in searching for the explant. In either case the position of the embryo was indicated by a circle made with a wax pencil upon the surface of the lid of the Petri dish. Sterile conditions were maintained during the course of the experiment.

The embryos of known age after coitus were isolated in cold or warm sterile saline solution (0.9%) and transferred to the surface of the medium, together with some of the saline solutions, by means of a pipette. After being spread out, they were incubated at $37^1/_2$°C. for 24–96 hours. The media employed were of an opaque, milky appearance and, in order to see the white explants, a sufficient amount of lampblack was added since

illumination of the field had to be from above. Under the circumstances it was only possible to see that the embryo was spread flat. It was impossible to determine the orientation of the embryo or to know whether the dorsal or ventral surface was uppermost. After varying intervals the regions of the media containing the embryos were isolated and transferred to Bouin's fixing solution. In most cases the embryo floated from the surface of the media at once, but if not, the two could be separated easily after fixation.

In all, forty-five explantations were successfully made of which ten showed some growth, as indicated by the outwandering of cells from the periphery of the tissue mass, and six showed a slight degree of further differentiation. Many of the explants had either been injured during isolation and transfer, or had not been spread out evenly. These subsequently formed masses of undifferentiated and necrotic tissue in which any indication of embryonic organization had disappeared. Others, which had not apparently been injured, failed to differentiate further, and simply persisted for varying lengths of time before becoming necrotic. However, even after 96 hours the embryos retained their external form and there could be seen a clear separation between the embryonic area and the adjacent trophoblast. On sectioning these cases, the cellular structure was found to be markedly atypical and foci of degeneration numerous. There were no mitotic figures and apparently no cellular differentiation beyond the stage at explantation.

Of those explants which did show some further differentiation two had been cultured on medium number one, and four on medium number two which was very slightly acidic. Twenty-four hours after explantation the presomite stage had differentiated two pairs of somites in one case and one pair in the other. The neural plate was more extensive, the neural folds were rudimentary and the primitive streak shorter. Neither brain nor heart had formed. The late primitive streak stages developed as far as the presomite stage or one pair of somites. Considerable distortion had occurred in all cases as well as contraction of the embryonic area. Differentiation went no further.

These preliminary experiments would seem to indicate that with modifications of the medium it may be possible to secure more complete differentiation comparable to that obtained previously with chicken plasma plus chick embryo extract. The failure or retardation of development, outside of the mechanical factors involved, may be due in large part to the lack of a food supply which can be readily assimilated by the explant. The explant does not liquefy the medium.

[1] Contribution No. 7. I am indebted to the NATIONAL RESEARCH COUNCIL for financial assistance in this investigation and to Dr. C. W. Grant for preparing some of the media.

[2] C. H. Waddington and A. J. Waterman, *Jour. Anat.*, **67**, 355 (1933).
[3] A. J. Waterman, *Am. Jour. Anat.*, **53**, 317 (1933).

A DIPLOID FEMALE GAMETOPHYTE OF SPHAEROCARPOS[1]

BY CHARLES E. ALLEN

DEPARTMENT OF BOTANY, UNIVERSITY OF WISCONSIN

Communicated February 8, 1934

In March, 1929, plants of *Sphaerocarpos Donnellii* Aust. were received from Prof. J. E. Judson; he had found them growing in the greenhouses at the University of Florida, Gainesville. The gametophytes of both sexes appeared to be of substantially typical character. Many sporophytes, mature and immature, were present, their spores being uniformly adherent in tetrads.

In October and November, 1929, tetrads from 6 of these sporophytes were sown. All the offspring from the tetrads of 5 sporophytes, examined but once and not kept, appeared to be typical. Twenty-nine offspring of tetrads from the sixth sporophyte were transplanted separately on January 17, 1930. Nineteen of the sporelings developed into ♀, 10 into ♂ clones. With 2 exceptions, all were typical. One ♀ clone bore at times a larger proportion of involucral appendages than is characteristic of typical clones; it was classed, with other similar clones which have appeared independently at various times, as *appendiculate*. The involucres of a second female clone (30.1004) bore a much larger proportion of appendages, often so numerous as to make the involucres appear distorted. The presence of involucral appendages, always numerous although varying in proportion from branch to branch and from time to time, has continued to mark this clone and the sub-clones into which it has been divided. The clone, unique in this respect among all the plants of *Sphaerocarpos* that I have observed, is designated *superappendiculate*.

Clone 30.1004 has served (previous to the present season) in 18 matings, involving 13 different ♂ clones. Eleven matings, involving 10 ♂ clones, produced sporophytes. Spore tetrads (as well as 10 dyads and 2 triads) have been sown, derived from sporophytic offspring of 5 of these matings. None of the spores from 3 matings germinated. One tetrad from a fourth mating gave rise to a doubtful plant which formed no sex organs and died while still small. Germination of the spores from a fifth mating gave results summarized below.

Mating, 30.1004 × 23.5058. Fertilized February 4, 1931. Mature sporophytes dried March 22–July 3. Nine dyads from 4 sporophytes

and 30 tetrads from a fifth sown September 14. Three dyads yielded 5 sporelings of which 3 were ♀ and 2 died before their sex could be determined. Twenty-five tetrads yielded sporelings (2–4 each); the sporelings from 2 of these tetrads died before their sex was determined; the other 23 tetrads produced offspring as follows:

4 tetrads yielded (each) 2 ♀ ♀, 2 ♂ ♂.
2 tetrads yielded (each) 2 ♀ ♀, 1 ♂.
6 tetrads yielded (each) 4 ♀ ♀.
5 tetrads yielded (each) 3 ♀ ♀.
3 tetrads yielded (each) 2 ♀ ♀.
3 tetrads yielded (each) 1 ♀.

Since the development of 2 ♀ and 2 ♂ offspring from the spores of a tetrad is a rule to which no certain exception has previously been observed in *Sphaerocarpos*, the occurrence in this case of at least 11 out of 23 tetrads with either 3 or 4 female-producing spores each demanded an explanation. The facts (although not as yet conclusive on this point) suggest that the tetrads of the family may have been of 2 classes: each tetrad of one class consisting of 2 male- and 2 female-producing, each of the other class of 4 female-producing spores.

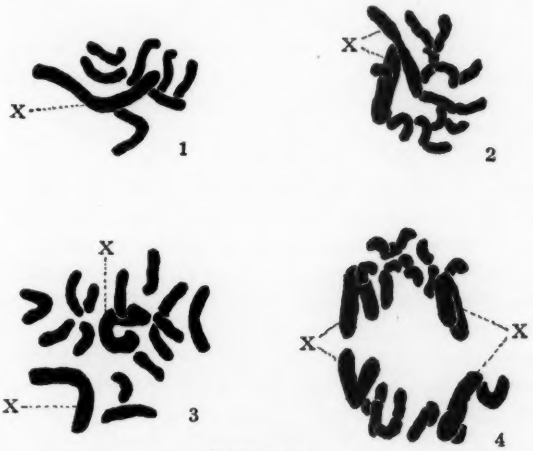
The usual 2–2 distribution results from the presence in the chromosome complement of the sporophyte of one *X*- and one *Y*-chromosome, the *X* passing to 2 of the spores of each tetrad, the *Y* to the other 2. In the present case, the simplest assumption seemed to be that the sporophyte producing the spore tetrads possessed more than one *X*-chromosome. If so, the additional *X* presumably came from one of the gametophytic parents. The male parent (clone 23.5058) has died since the mating was made, so was not available for examination. The possibility of its having borne an *X*-chromosome (in addition to a *Y*) was virtually out of the question, since diploid gametophytes possessing both *X* and *Y* are intersexual and externally apparently female.[2] It remained, therefore, to examine the maternal clone (30.1004), which is still available in culture.

Some of the results of a cytological study of this clone are shown in figures 2, 3 and 4. Figure 3 shows an equatorial plate containing clearly 16 chromosomes, among them 2 large *X*'s. This may be compared with figure 1, representing the characteristic haploid chromosome complement of a ♀ plant—8, including 1 *X*. In figure 2 is shown a late prophase from clone 30.1004, with about 16 chromosomes, including 2 *X*'s. Figure 4 shows an anaphase (not all the autosomes appearing in the section); there are 2 pairs of sister *X*'s, lagging somewhat behind the autosomes—a behavior characteristic of the *X* in haploid females.

Clone 30.1004 is therefore diploid, with 2 *X*-chromosomes and 14 autosomes. In a paper now in press[2] a diploid ♂ with 2 *Y*'s, and diploid

intersexes each with an X and a Y, have been reported. The present case seems to complete the list of eudiploid possibilities in this species. It may be added that sections from plants of this clone have shown numerous archegonia but nothing resembling the intersexual organs borne by a diploid plant with 1 X- and 1 Y-chromosome.

It seems clear that the sex chromosomes (2 X and 1 Y) present in the (presumably) triploid sporophyte which gave rise to the family here discussed were differently distributed in the course of meiosis in different



FIGURES 1-4

Figure 1. Chromosome complement characteristic of haploid ♀ gametophytes of *Sphaerocarpos Donnellii:* 1 X, 7 autosomes. Equatorial plate from clone R35G. Figure 2. Late prophase, diploid clone 30.1004:2 X's, ca. 14 autosomes. Figure 3. Equatorial plate, same clone: 2 X's, 14 autosomes. Figure 4. Anaphase, same clone: 2 lagging X's in each daughter group. All ca. 4000 ×.

spore mother cells. An indication of the nature of chromosome-segregation in this triploid sporophyte may be gained by a study of such of its gametophytic progeny as still survive. In the light of previous results it is probable that some of the unexpectedly numerous (apparent) females in this family will prove to be intersexes with both X- and Y-chromosomes.

Nothing can be said at present regarding the transmission of the superappendiculate character. Observations on members of the family here in question have thus far given conflicting and confusing results. The confusion may be due to the necessarily unequal or irregular meiotic distribution of chromosomes.

[1] Work done with the aid of grants from the University of Wisconsin Research Fund and the Wisconsin Alumni Research Foundation. At various periods during the progress of this study, assistance has been rendered by Ineva F. Reilly, Dorothy C. Bauch, Margaret B. Siler and Fred W. Tinney. Most of the cytological preparations were made by Donald W. Chamberlain under a CWA appointment.

[2] Allen, C. E., "The Occurrence of Polyploidy in Sphaerocarpos," *Amer. Jour. Bot.* (in press).

A SECOND EOCENE PRIMATE FROM CALIFORNIA

By Chester Stock

Balch Graduate School of the Geological Sciences, California Institute of Technology

Read before the Academy, Wednesday, November 22, 1933

Introduction.—In several papers, published in the PROCEEDINGS,[1] I have described some of the very late Eocene mammals obtained by the California Institute at Locality 150 in the Sespe deposits of southern California. Among the forms recently recorded[2] is the tarsiid, *Chumashius balchi*, related to the genera *Omomys* and *Hemiacodon*. The present paper deals with a second primate, evidently related to and presumably derived from the Bridger genus *Washakius*. This type and associated mammals were found at Locality 180, stratigraphically lower in the Sespe deposits than Locality 150. The faunal stage represented here is clearly earlier than that now being recorded from the latter locality, and is presumably upper Eocene in age. In view of the interest which attaches to fossil members of the primate group in North America, the new type is described in advance of a fuller statement of the occurrence of this older Eocene fauna in the Sespe, north of the Simi Valley.

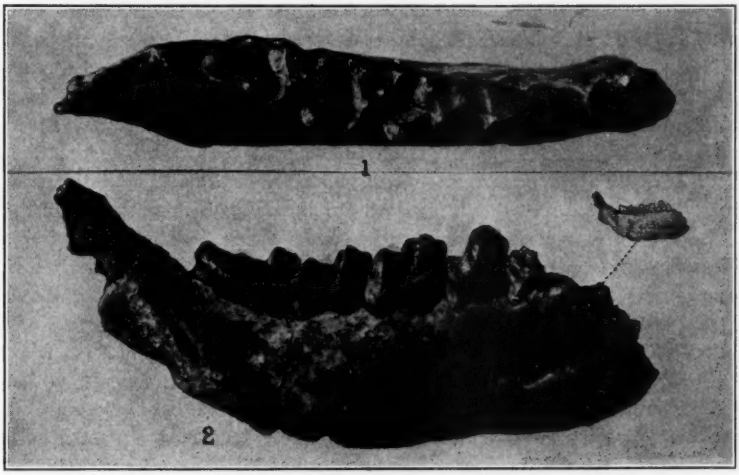
Dyseolemur pacificus, n. gen. and n. sp.

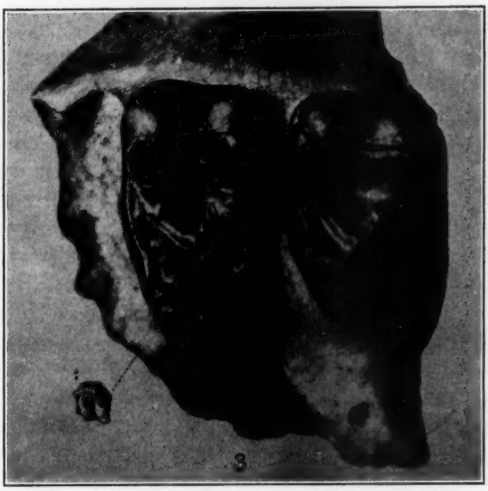
Type Specimen.—No. 1395 Calif. Inst. Tech. Vert. Pale. Coll., a fragmentary right ramus with cheek-tooth series $P\bar{4}$-$M\bar{3}$ complete, a small portion of $P\bar{3}$ and alveoli for $P\bar{2}$, C, and one or two incisors, plate 1, figures 1 and 2.

Paratype.—No. 1528, a maxillary fragment with $M\underline{2}$ and $M\underline{3}$, plate 1, figure 3.

Referred Specimens.—Several fragmentary rami with cheek-teeth, Nos. 1529, 1530 and 1531.

Locality.—Tapo Ranch, Locality 180 Calif. Inst. Tech. Vert. Pale., approximately 3 miles east of Locality 150 and approximately 700 feet lower in the Sespe deposits, north of the Simi Valley, Ventura County, California. Upper Eocene.





EXPLANATION OF PLATE

Dyseolemur pacificus, n. gen. and n. sp.

Figures 1 and 2, type specimen, No. 1395, occlusal and lateral views; × 6.

Figure 3, paratype, No. 1528, occlusal view; approximately × 10. Smaller figures represent specimens in natural size.

California Institute of Technology Collections. Sespe Upper Eocene, California.

Generic and Specific Characters.—$M\bar{3}$ more reduced in size than comparable tooth in *Washakius*. Heel less expanded transversely than in Bridger genus. Posterior portion of heel does not form a distinct lobe nor does it possess two distinct cusps (double hypoconulid) as in *Washakius*. Molars with small metastylids. $P\bar{4}$ subtriangular rather than quadrate in cross-section. Smaller than *Washakius insignis*.

Description.—The present type from the Sespe exhibits in shape of lower jaw, presence of metastylid on the lower molars and wrinkled enamel of the teeth, as well as in the features of the upper teeth in the paratype, a series of characters in which it differs decidedly from *Chumashius balchi* Stock. It is clearly evident also from the field occurrence that *Dyseolemur* is found distinctly lower than *Chumashius* in the Sespe beds, as exposed north of the Simi Valley, California. In presence of metastylid in the lower molar teeth, *Dyseolemur* is more like *Washakius* and *Shoshonius* than like other recorded anaptomorphs from the Eocene basins of the Rocky Mountain Province. Nearest resemblance appears to be with the Bridger genus *Washakius*, with which it agrees in shape of lower jaw and in characters of the lower and upper dentition. *Shoshonius* is a distinctly earlier form, in which the upper molars differ from those of the paratype of *Dyseolemur* in presence of a mesostyle.

The jaw, No. 1395, deepens noticeably in the symphyseal region. A small mental foramen is situated beneath the anterior root of $P\bar{4}$. In front of this and separated from it by a bar of bone is an open canal which leads forward. The groove may lead into the foramen below the bar. In the type specimen the lower teeth anterior to $P\bar{4}$ are either absent or, as in the instance of $P\bar{3}$, but imperfectly preserved. Anterior to the alveolus for the third premolar can be seen the socket for the single-rooted $P\bar{2}$. Immediately in front of this is a larger alveolus for the canine. Remnants of the alveolar walls of at least one incisor can be discerned apparently in front of the socket for the canine. All of the alveoli are crowded. Thus, in number of teeth, behind the last incisor, *Dyseolemur* agrees with *Washakius*, *Chumashius*, *Hemiacodon* and *Omomys* and differs from *Anaptomorphus* and *Tetonius*.

$P\bar{4}$ possesses a basal cross-section of crown, which is subtriangular rather than quadrate. In this tooth the paraconid and metaconid do not form such distinct cusps as in *Washakius*. An external cingulum is clearly shown on the molars and fourth premolar, plate 1, figure 2. In the molars of *Dyseolemur* the metastylid is smaller than in comparable teeth of *Washakius*, although in some specimens of the latter genus the metastylid is relatively as small. The third lower molar is distinctly more reduced in size than in *Washakius*. The heel is less expanded transversely, with no tendency to form a well-defined posterior lobe as in the Bridger genus. The metastylid is small and the notch between it and the entoconid may be

acutely incised or may be open as in *Washakius*. In the Sespe type it is not a distinctly formed cusp as in *Washakius*, but, because of an expansion in a fore and aft direction, becomes crest-like. Posteriorly, the entoconid is connected by a ridge with the hypoconulid. In $M\bar{3}$ of the Bridger genus two distinct cusps (double hypoconulid) are present, whereas in our form only a single cusp is situated at the posterior end of the occlusal surface. A small ridge runs outward and then forward from the hypoconulid, meeting a ridge which extends backward from the hypocone. The difference between *Dyseolemur* and *Washakius* in the region of the talonid of $M\bar{3}$ is seen not only in the type, No. 1395, but also in the referred specimen, No. 1530. The trigonid portion of the crown in $M\bar{2}$ and $M\bar{3}$ of the Californian form is somewhat less compressed anteroposteriorly. The length (in millimeters), as measured from the anterior end of $P\bar{4}$ to the posterior end of $M\bar{3}$, in No. 1395 is 7.6. In a specimen of *Washakius insignis* (No. 13235–2 Yale Peabody Mus. Coll.) the comparable measurement is 9.1.

$M\underline{2}$ and $M\underline{3}$ are present in the paratype, No. 1528, plate 1, figure 3. The third molar is shorter transversely than $M\underline{2}$, but possesses practically the same anteroposterior diameter as in this tooth. These teeth are decidedly smaller than those of *Hemiacodon gracilis* (Nos. 12012, 12030 Amer. Mus. Nat. Hist. Coll.) from the Bridger. They are likewise smaller than the comparable teeth in *Washakius* (see measurements below). The cingulum is not so well developed in No. 1528 as in *Washakius insignis*. A mesostyle is absent. In $M\underline{2}$ the hypocone is situated posterior and slightly internal to the protocone, but does not have the cingular position seen in *Hemiacodon*. In the Californian specimen the summits of the protocone and hypocone are situated farther toward the outer side, with reference to the inner border, than in *Washakius insignis*. The intermediate cuspules are minute but discernible. $M\underline{3}$ is pointed internally. No hypocone is evident, but the postero-internal side of the tooth is abraded. A very small but distinct metaconule can be seen. In *Dyseolemur*, as in *Washakius*, the enamel of the upper molars is wrinkled, particularly, as noted by Wortman for the Bridger genus, that covering the inner portion of the crowns of these teeth. Of the two specimens of *Washakius*, figured by Wortman,[3] No. 1528 is more like No. 13235–4 Yale Peabody Mus. (Fig. 146) than like No. 13235–2 Yale Peabody Mus. (Fig. 145) in narrowness of $M\underline{2}$ and in the pointed inner end of $M\underline{3}$.

COMPARATIVE MEASUREMENTS (IN MILLIMETERS)

	NO. 1528 C.I.T. SESPE	NO. 13235–2 Y.P.M. BRIDGER	NO. 13235–4 Y.P.M. BRIDGER
Length, anterior end of $M\underline{2}$ to posterior end of $M\underline{3}$	3.7	4.2	4.1
$M\underline{2}$, transverse diameter	3.25	3.4	3.5
$M\underline{3}$, transverse diameter	2.8	3.1	3.15

[1] Stock, C., *Proc. Nat. Acad. Sci.*, **18**, 518–523 (1932); **19**, 434–440, 481–486, 762–767 (1933).

[2] Stock, C., *Ibid.*, **19**, 954–959 (1933).

[3] Wortman, J. L., *Am. Jour. Sci.*, ser. 4, **17**, 210, figures 145–146 (1904).

COINCIDENCE AND INTERFERENCE IN DROSOPHILA MELANOGASTER

BY MORTON D. SCHWEITZER

DEPARTMENT OF ZOÖLOGY, COLUMBIA UNIVERSITY

Communicated January 24, 1934

I. *Definitions.*—Crossing-over in any region of a chromosome decreases the crossovers in near-by regions. This fact, discovered early in the work with *Drosophila*[1] and interpreted consistently with the chiasmatype hypothesis,[2] was termed "interference."[3] The term "coincidence" was applied to the "ratio of observed double crossovers to the chance expectation." It is apparent that coincidence is an empiric measure of genetic data useful in predicting results of crosses, whereas interference refers to the interpretation of cytological events that may be inferred from the data.

II. *Analysis of Coincidence Data.*—The first extended analysis of coincidence data was made by Weinstein.[4] He found that coincidence of double crossovers for eosin-ruby was 1.0 with sable-forked, 0.9 with forked-fused and 0.7 with forked-cleft; i.e., there is a shortage of double crossover chromosomes for regions beyond sable-forked; (for distances nearer than eosin-ruby to sable-forked, coincidence was known to have shown progressively lower values, being zero for distances less than 15 map units to the right of eosin). This was interpreted to mean that the "section included between the two crossovers of a double tends to be of a particular length."[5] For the *X* chromosome, the modal length turned out to be about 46 map units. There are, however, two apparent objections to the experimental and analytical procedure here involved. (1) In all of the experiments cited above only four loci were used. There were, it turns out, numerous undetected double crossovers, and some singles were probably also undetected. The influence of these uncontrolled variables on the coincidence values was not known and could not have been predicted but can scarcely be disregarded. A further objection (2) can be urged, namely, the arbitrary analytical procedure of discarding the triple crossovers from his double crossover classes. His argument that another mechanism was involved is unsupported. Weinstein pointed out that no significant decline beyond 46 units is observed if his data are recalculated without omitting triples.

TABLE 1

COINCIDENCE VALUES FOR X CHROMOSOME

sc(1) (2)cv(3) (4) v(5) (6)f(7)

				ec		ct	g	bb			
REGION	A	B	REGION	A	B	REGION	A	B	REGION	A	B
1,2	0.0	...	2,3	0.0	0.1	6,5	0.2	0.1	7,6	...	0.2
1,3	0.1	...	2,4	0.3	0.3	6,4	0.6	0.8	7,5	...	0.6
1,4	0.4	...	2,5	0.9	0.8	6,3	0.8	1.0	7,4	...	0.7
1,5	0.9	...	2,6	1.0	1.0	6,2	1.0	1.0	7,3	...	0.9
1,6	0.9	...	2,7	...	1.1	6,1	0.9	...	7,2	...	1.1
1,7	...	...							7,1	...	...

A—Bridges and Olbrycht, 1926—20,786 flies; B—L. V. Morgan, 1933—2623 flies.

An analysis of more extensive data by Weinstein's method, omitting triples, on the same chromosome and with few undetected doubles[6] (table 1), shows:

(*a*) That coincidence of double crossovers rises as the distance between the two breaks increases (one being fixed in position) until it reaches its theoretical expectation, *but that it does not fall below the expected value beyond that region.*

(*b*) That the influence of undetected double crossovers affects very materially the coincidence values obtained. In table 2 the data of Bridges and Olbrycht are reclassified in a number of ways, each resulting in different percentages of undetected singles and doubles; two of Weinstein's crosses are included for comparison.

TABLE 2

BRIDGES AND OLBRYCHT RECALCULATED

	CROSS	REGION UNMARKED	% UNDETECTED DOUBLES*	COINCIDENCE BETWEEN EXTREME REGIONS
1.	sc-ec-cv-ct-v-g-f	...	0.0%	1.0
2.	sc-ec-cv----v-g-f	cv-v	0.2	1.0
3.	---ec-cv----v---f	cv-v v-f	0.2 0.3	1.0
4.	sc-ec-------v---f	ec-v v-f	0.8 0.3	1.0
5.	---ec-cv------g-f	cv-g	1.2	0.7
6.	sc-ec---------g--	ec-g	2.6	0.4
7.	---ec---------g-f	ec-g	2.6	0.7
8.	sc----------g-f	sc-g	3.9	0.6
9.	sc-ec----------f	ec-f	5.8	0.5
	Weinstein			
1.	we-rb-------s-f	rb-s	1.0	1.0
2.	we-rb---------f-cl	rb-f	4.5+	0.7

* (6) p. 54.

It is obvious from this table that for regions widely separated, large percentages of undetected doubles are associated with coincidence values smaller than 1.0. It is also apparent that the data of Weinstein are approximated by this type of treatment. These findings are advanced in support of the objections previously urged to the older analyses.

It might have been expected that comparable data for the autosomes would yield similar results.

TABLE 3

COINCIDENCE VALUES FOR II CHROMOSOME

		al(1) tx (2) b	(3) pr	(4) cn	(5) vg	(6)L(7)sp	
1,2–0.1	2,1–0.1	3,1–0.4	4,1–0.6	5,1–0.7	6,1–0.8	7,1–0.7	
1,3–0.4	2,3–0.1	3,2–0.1	4,2–0.5	5,2–0.9	6,2–0.8	7,2–0.9	
1,4–0.6	2,4–0.5	3,4–0.4	4,3–0.4	5,3–0.7	6,3–0.7	7,3–0.9	
1,5–0.7	2,5–0.9	3,5–0.7	4,5–0.2	5,4–0.2	6,4–0.3	7,4–0.4	
1,6–0.8	2,6–0.8	3,6–0.7	4,6–0.3	5,6–0.2	6,5–0.2	7,5–0.3	
1,7–0.7	2,7–0.9	3,7–0.9	4,7–0.4	5,7–0.3	6,7–0.1	7,6–0.1	

A recalculation of some comparable data from the II chromosome,[7] given in table 3, shows that the same relation does not hold in its simple form. This table shows that coincidence increases with distance but does not reach 1.0, even for sections not situated in the same arm. On the other hand, *it has been repeatedly demonstrated*[8] *that crossing-over in one arm of the autosomes of Drosophila is completely independent of the other arm* with the exception of regions close to the spindle fibre. The contradiction suggests the presence of an error in the theoretical derivation of coincidence.

III. *Calculation and Interpretation of Interference.*—If interference has meaning in terms of the chromosomal mechanism it must refer to the influences simultaneously operating throughout the chromosome; a general mathematical solution of the problem is not yet possible. A simple formulation, however, may be made for the effect of crossing-over in one region on crossing-over in another, *when all other regions stay intact*, as shown in table 4.

TABLE 4*a*

RAW DATA OF BRIDGES AND OLBRYCHT

0–9554	6 –1525	2,3– 8	3,6 –161	1,3,6–3	2,4,6 –4	1 2,3,6–1
1– 994	1,2– 3	2,4– 95	4,5 – 76	1,4,5–7	2,5,6 –1	
2–1465	1 3– 10	2,5–191	4,6 –225	1,4,6–9	3,4,6 –4	Total—20,786
3–1425	1,4– 87	2,6–230	5,6 – 46	1,5,6–1	3,5,6 –2	
4–2512	1,5– 140	3,4– 26	1,2,6– 1	2,3,4–1	4,5,6 –1	
5–1717	1,6– 147	3,5–110	1,3,5– 2	2,4,5–1	1,2,3,4–1	

TABLE 4*b*

SAMPLE CALCULATION OF FRACTION REALIZED

0	1	4	1.4	Total—13,147
9554	994	2512	87	Fraction realized $\frac{87 \times 13{,}147}{1081 \times 2599} = 0.4$
	87	87		
	1081	2599		

The ratio of double crossovers obtained over expected, under these conditions, may be termed the "fraction realized," and is shown for chromosomes X and II in tables 5 and 6.

TABLE 5

FRACTION REALIZED—RECALCULATED FOR DATA OF TABLE 1

B&O	LVM	B&O	LVM	B&O	LVM	B&O	LVM	B&O	LVM	B&O	LVM	B&O	LVM
1,2	0.0 ...	2,1	0.0 ...	3,1	0.1 ...	4,1	0.4 ...	5,1	0.8 ...	6,1	1.0 ...	7,1	... ...
1,3	0.1 ...	2,3	0.0 0.2	3,2	0.0 0.2	4,2	0.3 0.4	5,2	0.9 0.9	6,2	1.0 1.0	7,2	... 1.2
1,4	0.4 ...	2,4	0.3 0.4	3,4	0.1 0.2	4,3	0.1 0.2	5,3	0.5 0.8	6,3	0.8 1.0	7,3	... 1.2
1,5	0.8 ...	2,5	0.9 0.9	3,5	0.5 0.8	4,5	0.3 0.4	5,4	0.3 0.4	6,4	0.6 0.8	7,4	... 0.8
1,6	1.0 ...	2,6	1.0 1.0	3,6	0.8 1.0	4,6	0.6 0.8	5,6	0.2 0.1	6,5	0.2 0.1	7,5	... 0.1
1,7	... ...	2,7	... 1.2	3,7	... 1.2	4,7	... 0.8	5,7	... 0.1	6,7	... 0.2	7,6	... 0.2

TABLE 6

FRACTION REALIZED—RECALCULATED FOR TABLE 3

1,2–0.2	2,1–0.2	3,1–0.5	4,1–1.0	5,1–1.0	6 1–1.0	7,1–0.9
1,3–0.5	2,3–0.2	3,2–0.2	4,2–0.8	5,2–0.9	6,2–1.0	7,2–1.1
1,4–1.0	2,4–0.8	3,4–0.7	4,3–0.7	5,3–1.1	6,3–1.0	7,3–1.0
1,5–1.0	2,5–1.2	3,5–1.1	4,5–0.4	5 4–0.4	6,4–0.5	7,4–0.7
1,6–1.0	2,6–0.9	3,6–1.0	4,6–0.5	5,6–0.4	6,5–0.4	7,5–0.6
1,7–0.9	2,7–1.1	3,7–1.0	4,7–0.7	5,7–0.6	6,7–0.4	7,6–0.4

These tables show for both chromosomes that interference disappears at some distance and *never* reappears. Table 6 also shows that the two arms of the II chromosome are independent except for regions near the spindle fibre (near purple). Similar calculations on less comprehensive data for chromosome III[9] yield results which fit the interpretations here given and will be more fully treated in another place.

Since this method of calculation, restricted to chromosomes with two or fewer crossovers, is able to reconcile data hitherto contradictory, it has a practical superiority over the older methods of calculation. It has a theoretical advantage as well. It will be recalled that Weinstein argued for the elimination of 1, 2, 3 crossovers from coincidence calculations of 1, 3 because different conditions were involved. It is only an extension of his argument to point out that, since a crossover in any region is affected by the crossover relations in most others, the mutual influence of pairs of regions is

significantly expressed only when other regions are not altering this influence. The "fraction realized" then is a direct measure of the uncomplicated influence of one region upon another.

IV. *Discussion.*—It is apparent from what has already been said that the validity of coincidence as ordinarily computed remains unchanged when it is desired simply to predict the frequency of particular classes in an experiment involving previously tested loci. For that purpose, the method of calculating interference here used, although related to coincidence, has not yet been adapted. But the ordinary calculation of coincidence between *non-adjacent* regions has little theoretical significance and, as has already been shown in the case of the II chromosome, may lead to conclusions inconsistent with other data; these anomalies will disappear if interference is calculated as here suggested. Thus, different methods of calculation for genetic prediction and cytological interpretation must be used.

There remains to be given an interpretation for the findings that interference does not reappear once it is reduced to zero. It seems altogether likely that there is no modal length, and that all lengths above a certain minimum (which fall in the region giving interference) are equally likely to occur. A comprehensive analysis of this and other problems raised must await the completion of experiments involving chromosomes completely marked.

Note.—Graubard[10] has used the calculation of fraction realized from my table 6 to verify his conclusion concerning the independence of the two arms of chromosome II. The present writer does not concur in the interpretation there given from the computations on the model of my table 3.

[1] Sturtevant, A. H., *Jour. Exp. Zoöl.*, **14**, 43–59 (1913).

[2] Morgan, T. H., Sturtevant, A. H., Muller, H. J., and Bridges, C. B., *The Mechanism of Mendelian Heredity*, New York, 1–262 (1915).

[3] Muller, H. J., *Amer. Nat.*, **50**, 193–221, 284–305, 350–366, 421–434 (1916).

[4] Weinstein, A., *Gen.*, **3**, 135–173 (1918).

[5] Morgan, T. H., Bridges, C. B., and Sturtevant, A. H., *Bibliogr. Gen.*, **2**, 1–263 (1925).

[6] Bridges, C. B., and Olbrycht, T. M., *Gen.*, **17**, 41–56 (1926).

[7] Morgan, L. V., *Gen.*, **18**, 250–283 (1933); Graubard, M. A., *Gen.*, **17**, 81–105 (1932).

[8] Sturtevant, A. H., *Biol. Zlb.*, **46**, 697–702 (1926); *Publ. Carnegie Inst. Wash.*, **421**, 1–27 (1931); Dobzhansky, Th., *Gen.*, **15**, 347–399 (1930); *Gen.*, **16**, 629–658 (1931); Payne, F., *Gen.*, **9**, 327–342 (1924); Graubard, M. A., *Gen.*, **17**, 81–105 (1932); Redfield, H., *Gen.*, **15**, 205–251 (1930); *Gen.*, **17**, 137–152 (1932).

[9] Bridges, C. B., *Publ. Carnegie Inst. Wash.*, **399**, 63–89 (1929); Dobzhansky, Th., *Gen.*, **15**, 347–399 (1930)

[10] Graubard M. A., *Gen.*, **19** 93–94 (1934).

THE RÔLE OF THE "CHROMOSOME SHEATH" IN MITOSIS, AND ITS POSSIBLE RELATION TO PHENOMENA OF MUTATION

By C. W. METZ

DEPARTMENT OF EMBRYOLOGY, CARNEGIE INSTITUTION OF WASHINGTON, AND DEPARTMENT OF ZOÖLOGY, JOHNS HOPKINS UNIVERSITY

Communicated February 14, 1934

Until recently, cytologists who have considered chromosome movements and chromosome associations have given little attention to the possible rôle of the "chromosome sheath" or envelope—the transparent and apparently gelatinous layer of material which may be seen under favorable conditions surrounding the stained chromosome. This neglect has been due, presumably, to the fact that in most cases such a structure is not visible and that in consequence there is no certainty that the "sheath" is characteristic of chromosomes in general. There seem to be good grounds, however, for considering that the sheath is such a typical structure, and also for considering that it may play an important rôle in the activities of the chromosomes, particularly during mitosis. Such consideration, furthermore, when taken in connection with recent evidence concerning the artificial induction of mutations, suggests that the chromosome sheath may likewise have a direct connection with the phenomena of mutation.

It is the purpose of the present paper to note very briefly some of the evidence on these points. Detailed treatment will be reserved for another paper.

As regards direct observation of the "sheath" there is, on the one hand, strong evidence that wherever conditions are suitable for revealing the presence of such a structure it is visible. The nature of this evidence has been noted in earlier papers[1] and need not be considered here. On the other hand, it is clear that commonly the sheath would not be visible because of the fact that the chromosomes lie in a medium (nucleoplasm or spindle substance) which, like the sheath, remains transparent even in stained preparations.

The evidence indicates that the sheath is a layer of differentiated material around the chromosome, fairly definitely delimited, but without any peripheral membrane or cortex. Presumably it is a product of the activity of the chromosome proper[2] and is characterized in part by its physical consistency. Perhaps it may even be continuous with, and similar to, the matrix of the chromosome proper. It appears to be present not only when the chromosomes are condensed, as during mitosis, but also when they are elongate and thread-like (Metz and Nonidez, loc. cit.).

Details as to the nature and appearance of the structure are not, however, of primary concern in the present connection. It is desired here to note particularly certain characteristics of chromosome behavior which indicate the presence of the sheath and the rôle which it may play even when it is not directly visible.

1. *The Rôle of the Sheath in Mitosis.*—Orientation of the chromosomes on the metaphase plate in mitosis, involving a definite spacing of the chromosomes, usually approximately equidistant from one another, is, in the writer's opinion, best accounted for on the assumption that the stainable part, or chromosome proper, is surrounded by a differentiated, gelatinous layer which keeps it from coming into contact with the other members. This feature has been elaborated somewhat in an earlier paper (Metz '28, loc. cit.).

Likewise, certain features of chromosome behavior, from prophase through early anaphase, which have never received adequate explanation, are subject to ready interpretation if the presence of a sheath is taken into consideration. It is well known that during late prophase and metaphase in many forms each chromosome is split along its entire length, with the two daughter halves distinctly separated from one another throughout. Yet these daughter halves may remain accurately aligned and approximately equidistant from one another, even in the case of long chromosomes and during a period of time in which the chromosomes undergo considerable movement.

The persistence of this accurate alignment has been attributed to an hypothetical "attraction" between the daughter halves, although it has been recognized that in other respects chromosome behavior during this period indicates a repulsion rather than attraction between the daughter halves. It is clear that the original division or splitting of the chromosome is an autonomous act, and that some sort of repulsion serves to effect a slight separation of the daughter halves. Their failure subsequently to fall apart, or become more widely separated, during prophase and metaphase movements, and also the relative absence of twisting and other irregularities of alignment are not only understandable on the assumption of an enveloping sheath, but serve in themselves to suggest the presence of such a structure.

Even more striking evidence is provided by the subsequent behavior of the elements in question. It is seen most clearly, perhaps, in the case of long chromosomes which at metaphase extend some distance across the equator of the spindle or even out of it. In some organisms, as the poleward movement of the daughter halves of such chromosomes progresses, beginning at the "spindle fibre" locus, the point of divergence is sharply delimited and each half is bent almost at right angles at this point, the unseparated arms remaining in close association up to the point of

divergence. In such cases the point of divergence between the two halves may remain approximately fixed within the spindle while the arms move in toward it as separation progresses.

Such behavior receives a simple explanation if it is assumed that the enveloping sheath serves at first to hold the two sister chromosomes together and that it gradually undergoes longitudinal division as separation progresses, beginning at the spindle fibre locus. Variations in the degree of association and in the behavior of the sister halves in different organisms would, on this view, depend on variations in the time and rate of division of the sheath. The same principle may be applied to homologues in meiosis.

Evidence has been presented in another paper[3] which suggests that the chromosome sheath may take an active part in causing the movement of the chromosome toward the pole in anaphase, after separation of the daughter halves has taken place, and that to this extent the movement of the chromosome may be autonomous. A somewhat similar suggestion has been made in earlier papers by Bleier (loc. cit.) but on his view, as noted above, the material enveloping the chromosome is independent of the chromosome itself in both origin and activity.

If the above considerations are valid it would appear that the chromosome sheath plays an important rôle throughout the whole process of mitosis (including meiotic divisions).

2. *Possible Relation of the Sheath to Mutation.*—On the assumption that the sheath is a characteristic structural component of chromosomes it seems probable from the available evidence that one of its primary functions is that of insulating the chromosome proper and preventing it from coming into direct contact with other formed bodies in the cell, including the other chromosomes.[4] Indications of this are seen not only in the spacing of the chromosomes on the equatorial plate, as noted above, but in their relations at all other stages. It would appear, therefore, that anything tending to disperse or otherwise alter the chromosome sheath might readily lessen or eliminate its insulating properties and permit the chromosomes to come into direct contact with one another. This consideration suggests a possible important relation of the sheath to mutation phenomena.

The well-known studies of Muller, and subsequently of many others, on the artificial induction of mutations, have shown clearly that irradiation with x-rays or radium greatly increases the rate of mutation in numerous organisms. Likewise, it greatly increases the frequency of occurrence of breaks in chromosomes, and of translocations, inversions and "reduplications" of parts of chromosomes. By comparing the incidence of gene mutations with that of the gross changes (translocations, etc.) Muller has secured evidence leading to the view[5] that the underlying process

which results in the former is essentially the same as, or bears a significant resemblance to, that which results in the latter type of change. Furthermore, since the gross changes are evidently not the result of direct modification of the molecular structure, it is inferred, on this view, that gene mutations may not be the result of such direct modification, but may be due to intimate contact and interaction between chromosomes such as does not occur frequently under ordinary conditions. "The irradiation has somehow (possibly by de-charging them) done away with the repulsion which normally holds chromosome strands apart from one another. . ." (Muller, loc. cit., p. 218).

Taken in connection with the considerations presented above, these findings suggest immediately that irradiation serves to disturb the normal insulating properties of the chromosome sheaths and permits intimate contacts between the chromosomes. It seems possible that such contacts may be a primary cause of mutation, particularly when they occur at a stage in which the chromosomes are long and thread-like. Such an interpretation would apply similarly to the effects of high temperature, which has also been shown to stimulate mutation.[6]

The interpretation just outlined is in harmony with the observed effects of irradiation and heat on chromosomes. Several investigators have shown[7] that these agents cause a clumping of chromosomes, often accompanied by fusion and subsequent fragmentation of some members. The observations indicate, at least in the case of irradiation with x-rays or radium, that a decrease in protoplasmic viscosity is produced.[8] On the present view this serves to solate the gelatinous "sheath" permitting chromosomes to come into contact with one another and to interact in such a way as to cause segmental rearrangements and gene mutations.

If this interpretation is correct it follows that any agent reducing the insulating properties of the chromosome sheath in this manner might increase the rate of mutation. Muller (loc. cit.) has suggested that if the cause of gene mutations is essentially the same as that of translocations, etc., it might be expected that other agents, including chemicals, would be effective in increasing the mutation rate. The present observations tend to support such an inference and to suggest the mechanism which may underlie the phenomena.

[1] Metz and Nonidez, *Biol. Bull.*, **46**, 153 (1924); Metz, C. W., *Genetics*, **10**, 345 (1925); Bleier, H., *La Cellule*, **40**, 82 (1930); *Genetica*, **13**, 27 (1931); Koerperich, J., *La Cellule*, **39**, 309 (1930); Schrader, F., *Zeits. f. wiss. Zool.*, **142**, 520 (1932).

[2] Bleier (loc. cit.) considers it as having an independent existence, but the evidence for this view does not appear convincing. See also Schrader, loc. cit.

[3] Metz, C. W., *Biol. Bull.*, **64**, 333 (1933).

[4] It does not, however, act merely as an inert layer. It must undergo numerous changes and it apparently has the power of selective action, as shown, e.g., by the fact that it does not prevent synapsis of homologous chromosomes.

[5] Muller, H. J., *Proc. VI Int. Cong. Genet.*, **1**, 213 (1932).

[6] See, e.g., Goldschmidt, R., *Biol. Zbl.*, **49**, 437 (1929); Jollos, V., *Ibid.*, **50**, 541 (1930), *Naturwis.*, **19**, 171 (1931), *Ibid.*, **21**, 455 (1933) and **21**, 831 (1933); Rokitzky, P. T., *Biol. Zbl.*, **50**, 554 (1930); Plough and Ives, *Proc. VI Int. Cong. Genet.*, **2**, 156 (1932); Grossman and Smith, *Amer. Nat.*, **67**, 429 (1933).

[7] See, e.g., Koernicke, M., *Ber. d. Deuts. Bot. Gesel.*, **23**, 404 (1905); Takimine, N., *Bot. Mag. Tokyo*, **37**, 109 (1923); Packard, C., *Quart. Rev. Biol.*, **6**, 253 (1931); Helwig, E. R., *J. Morph.*, **55**, 265 (1933); Lewis, M. R., *Arch. f. exp. Zellf.*, **14**, 464 (1933).

[8] Williams, M., *Ann. Bot.*, **39**, 547 (1925); Helwig, loc. cit. (1933).

---

*DEVELOPMENT OF THE EMBRYO SAC OF LILIUM HENRYI**

BY D. C. COOPER

DEPARTMENT OF GENETICS, THE UNIVERSITY OF WISCONSIN

Communicated January 29, 1934

While making a study of cell-plate formation in the macrogametophyte of *Lilium Henryi* Baker, the writer observed that the type of embryo-sac development was at variance with the so-called "lily type." Instead of the egg being removed from the macrospore mother cell by three divisions, four divisions actually intervene.

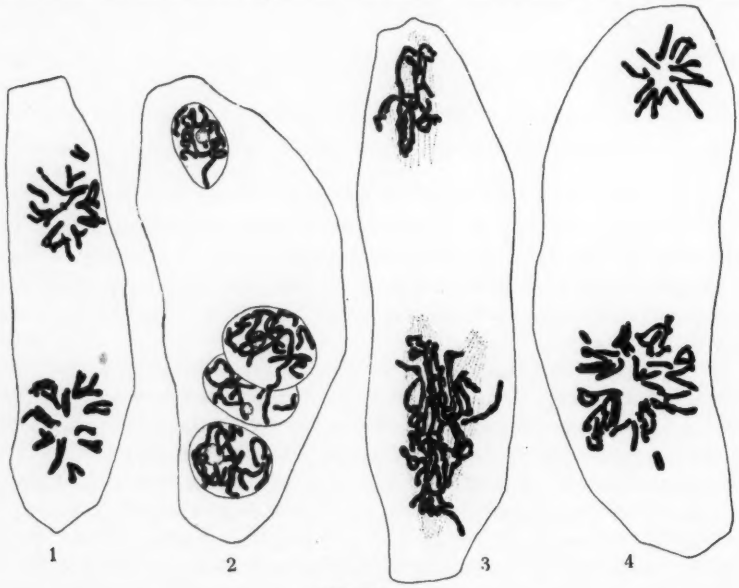
The nucleus of the macrospore mother cell and its daughter nuclei pass through the heterotypic and homoeotypic divisions without cell division so that a four-nucleate embryo sac is formed. A polar view of the homoeotypic equatorial plates shows 12 chromosomes on each spindle (Fig. 1). Three of the four nuclei formed as the result of the homoeotypic division pass to the chalazal end of the embryo sac (Fig. 2), and then all four nuclei divide simultaneously. During this division the spindles of the three chalazal nuclei unite. A multipolar spindle is evident at relatively early stages (Fig. 3), but ultimately a single bipolar spindle is formed and the 36 chromosomes becomes arranged on a common equatorial plate. In figure 4 both equatorial plates are visible in polar view. Twelve chromosomes can be counted on the spindle at the micropylar end of the embryo sac, whereas there are 36 on the chalazal spindle.

As a result of this division a four-nucleate embryo sac is again formed. Two small nuclei are near its micropylar end and two larger nuclei at the chalazal end. Earlier investigators (Sargant,[1] Mottier[2]) explained the greater size of the two chalazal nuclei as probably due to growth because this condition was found only in the larger embryo sacs. The spindles of the third division just referred to persist between the perinuclear zones which are formed about each daughter nucleus. The embryo sac elongates to some extent before the nuclei divide again.

Three of the four nuclei now present, the two at the micropylar end of the

embryo sac and one of the $3n$ nuclei in the chalazal region, undergo typical mitoses (Fig. 5). There is a much larger number of chromosomes on the chalazal spindle than on either spindle at the micropylar end of the sac. Mottier[2] observed the greater number of chromosomes on the chalazal spindle at this stage, but considered the greater number as due to an increased amount of chromatin material in consequence of growth.

The other $3n$ chalazal nucleus is at the tip of the embryo sac. It divides in a more or less abortive manner. Although there seems to be no regular



FIGURES 1–4

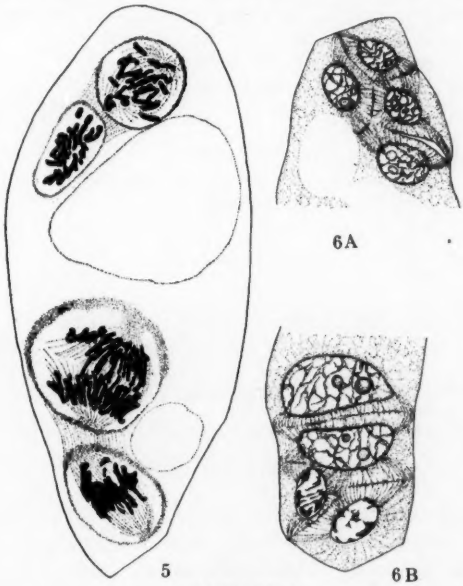
Figure 1. Polar view of the homoeotypic equatorial plates. Figure 2. Four-nucleate embryo sac. Three nuclei are in the chalazal region. Figure 3. Embryo sac showing fusion of the three chalazal spindles. Figure 4. Somewhat later stage than that of figure 3, showing polar view of equatorial plates.

formation of chromosomes, a typical spindle is formed and granules or masses of chromatic material pass to each pole.

The eight-nucleate embryo sac (Fig. 6) contains four small nuclei at the micropylar end (A) and two large and two smaller nuclei at the chalazal end (B). The two latter nuclei are abortive in appearance, a fact which is probably related to the early disintegration of two of the antipodal cells. Such abortive nuclei were also observed by Sargant[1] and Mottier.[2] The synergids and egg at the micropylar end and the antipodal cells at the chalazal end are formed as a result of cell-plate formation across the spin-

dles of the last division as well as across the persistent spindles of the preceding division. Two polar nuclei, one small (n) and one large ($3n$), are left in the primary endosperm cell.

A type of embryo-sac development similar to that herein reported has recently been found in other members of the Liliaceae. Bambacioni[3,4] described in detail the fusion of 3 of the 4 nuclei which had been formed as a result of the homoeotypic division in *Fritillaria persica*. These three nuclei come to lie in the chalazal end of the embryo sac and then divide simultaneously, the chromosomes becoming arranged on a common spindle.



FIGURES 5–6

Figure 5. Last nuclear division in the embryo sac.
Figure 6. Cell-plate formation. (*A*) Micropylar end of embryo sac. (*B*) Chalazal end.

Bambacioni and Giombini[5] found a similar fusion of 3 of the 4 nuclei in the 4-nucleate embryo sac of *Tulipa Gesneriana*. Bambacioni-Mezzetti[6] later examined three other members of the Liliaceae, namely, *Lilium candidum*, *L. bulbiferum* and *Tulipa praecox*. In all three species evidence was found for the fusion of 3 of the 4 nuclei formed as a result of the homoeotypic division. In the two latter species figures similar to my figure 4 are shown. In each case there are many more chromosomes on the spindle at the chalazal end of the embryo sac than there are on the one at the micropylar end.

When it was discovered that the development of the embryo sac in *Lilium Henryi* differs from the "lily type," the preparations of lily material used for class work in the Botany Department at the University of Wisconsin were examined. These slides include material from three species, *L. speciosum* Thunb., *L. philadelphicum* Linn. and *L. longiflorum* Thunb. var. *eximium*, Nichols (*L. Harrisii*, Carr.). The history of embryo-sac development in this material was found to be similar to that here described for *L. Henryi*.

* Papers from the Department of Botany and the Department of Genetics (No. 169), Agricultural Experiment Station, University of Wisconsin. Published with the approval of the Director of the Station.

[1] Sargant, E., *Annals Bot.*, **10**, 445–477 (1896).

[2] Mottier, D. M., *Jahr. Wiss. Bot.*, **31**, 125–158 (1898).

[3] Bambacioni, V., *Rend. Acc. Lincei Roma, Cl. Fis. Mat. Mat.*, Ser. **6.6**, 544–546 (1928).

[4] Bambacioni, V., *Annali Bot.*, **18**, 7–37 (1930).

[5] Bambacioni, V., and Giombini, A., *Ibid.*, **18**, 373–386 (1930).

[6] Bambacioni-Mezzetti, V., *Ibid.*, **19**, 365–382 (1932).

THE ULTRA-VIOLET ABSORPTION SPECTRUM OF CARBON SUBOXIDE GAS

By Richard M. Badger and Robert C. Barton

Gates Chemical Laboratory, California Institute of Technology, Pasadena

Communicated January 16, 1934

The carbon suboxide used in this investigation was prepared by the diacetyl tartaric anhydride method,[1] and was kindly supplied by Dr. W. E. Vaughn of this laboratory. The material was purified by four vacuum distillations and may have contained a trace of acetic acid as impurity which, however, would not have been disturbing in the region investigated.

The carbon suboxide gas was contained in a quartz absorption cell 15 cm. in length, provided with plane windows and a side arm containing an excess of the material in liquid form. By controlling the temperature of the latter the gas pressure in the cell could be regulated. The absorption cell itself was kept at room temperature, which was about 21°C. The carbon suboxide apparently kept well if sealed in perfectly clean quartz vessels, especially if maintained at low temperatures, but after long exposure to ultra-violet radiation a white deposit collected on the windows of the cell. This was probably a polymerized form of the suboxide and it quickly changed over to a reddish orange form if gently heated.

The light source used was a hydrogen discharge tube of special design

and the spectrograph was a large size Bausch and Lomb instrument of the Littrow type which gave a dispersion of about 4 Å per mm. at λ 3000. The spectrograms were taken on Eastman 40 plates. The wave-lengths were determined by comparison with the iron spectrum and with a few mercury

TABLE 1

ULTRA-VIOLET ABSORPTION MAXIMA OF CARBON SUBOXIDE

REGION		λ	ν	δν
	a	3197.9	31261	
1.	b	95.6	284	
	c	95.6	300	
	a	80.5	432	
2.	b	76.4	473	
	c	71.8	518	
	d	66.3	568	
	a	56.9	668	
3.	b	51.9	718	
	c	46.5	772	
	a	37.8	860	
4.	b	**30.3**	**937**	
	c	20.2	32040	
	a	13.9	105	
	b (?)	12.5	119	240
5.	c (?)	10.6	139	
	d	09.8	147	
	e	**06.9**	**177**	
	f	05.3	194	218
6.		**3086.0**	**395**	224
	a	**64.8**	**619**	
	b	63.3	635	
7.	c	61.8	650	
	d	60.4	666	
	a	49.9	778	
	b	48.2	796	244
	c	46.9	811	
	d	45.9	821	
8.	e	44.9	832	
	f	44.0	841	
	g	43.2	849	
	h	**42.0**	**863**	
	i	40.3	882	

REGION		λ	ν	δν
	a	3021.3	33089	238
9.	b	**20.2**	**101**	
	c	18.4	120	
	d	17.1	135	
	a	12.7	183	
	b	10.9	203	219
10.	c	09.2	218	
	d	07.2	244	
	e	**00.2**	**320**	
	f	2998.8	370	228
11.		**79.9**	**548**	
	a	62.0	751	233
12.	b	**59.4**	**781**	
	a	44.5	952	244
13.	b	39.8	34006	
	c	**38.2**	**025**	192
14.		**21.7**	**217**	
15.	a	07.0	390	
	b	01.1	460	
16.		**2887.9**	**617**	254
17.		**65.5**	**863**	193
18.		**51.7**	**35056**	

lines which were faintly present in the spectrum of the discharge tube.

Carbon suboxide was found to have a relatively strong absorption in a region the limits of which depend on the gas pressure or path length, but which may be given roughly as λ 3200 and λ 2500. The absorption consisted both of discrete bands which set in at about λ 3200 and could be

followed to λ 3050, and a continuum overlying the bands and extending considerably to the short wave side of λ 2537 with a maximum at about λ 2600. Owing to the great variation of absorption with wave-length only from five to seven bands are distinguishable on any given spectrogram, since further bands on the long wave side are too faint to appear and on the short wave side are masked by the strong continuum. By taking several exposures with different gas pressures eighteen bands, or regions of absorption maxima were obtained with sufficient intensity to measure.

The results of the investigation are best shown in figure 1 in which are reproduced microphotometer curves of spectrograms taken at five different gas pressures. The exact pressures are unknown but they are the vapor pressures ot the liquid at the temperatures indicated on the curves. The absorption maxima which were measured are indicated in the figure and the corresponding wave-lengths and frequencies are given in table 1. Some

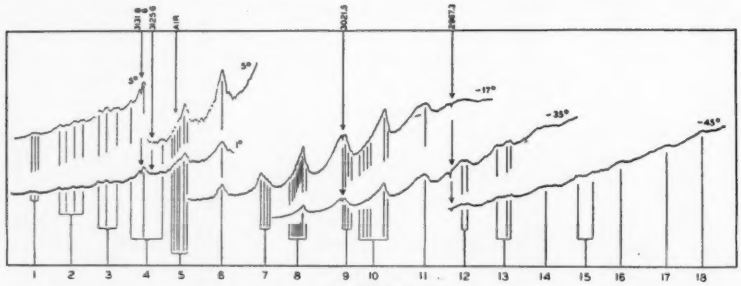


FIGURE 1

Absorption maxima in the ultra-violet spectrum of carbon suboxide gas.

mercury lines which appeared weakly in the spectrum of the discharge tube are indicated and it should be noted that on one of the plates there are a few weak air lines between the bands designated as 4 and 5, which appear as minima on the photometer curve. The continuous spectrum of the discharge was otherwise free from impurities.

Most of the bands show some evidence of complexity which can hardly be a rotational structure in the ordinary sense owing to the large moment of inertia of the molecule. Electron diffraction experiments on carbon suboxide[2] have yielded for the C—C distance 1.30 Å, and for the C—O distance 1.20 Å, and if the molecular is linear, as seems most probable, the moment of inertia perpendicular to the figure axis should be about 397×10^{-40}. The separation of the rotation lines should accordingly be about 0.14 $cm.^{-1}$, or 0.013 Å, which could not have been resolved by the instrument at our disposal.

Most of the absorption maxima seem to belong to a band progression with a spacing which averages about 223 frequency units, though on the

long wave side it appears that there may be some additional bands. The magnitude of this spacing indicates that we have to do with one of the transverse or "deformation" oscillations of the molecule, and the complexity of the bands is evidently due to the existence of several vibrational levels having nearly the same energy. In the case of the ideal linear molecule with Hooke's law binding the energy levels corresponding to the transverse vibrations are degenerate, but in the actual molecule this degeneracy is of course at least partly removed and there will be several transitions which will give rise to bands in nearly the same position in the spectrum. A somewhat similar complexity of structure has been observed in the ultraviolet bands of cyanogen.[3]

[1] Hurd and Pilgrim, *J. Am. Chem. Soc.*, **55**, 757 (1933).
[2] L. O. Brockway and L. Pauling, *Proc. Nat. Acad. Sci.*, **19**, 860 (1933).
[3] Sho-Chow Woo and Richard M. Badger, *Phys. Rev.*, **39** (1932).

EFFECT OF INHOMOGENEITY ON COSMOLOGICAL MODELS

BY RICHARD C. TOLMAN

NORMAN BRIDGE LABORATORY OF PHYSICS, CALIFORNIA INSTITUTE OF TECHNOLOGY

Communicated February 12, 1934

1. *Introduction.*—In the application of relativistic mechanics and relativistic thermodynamics to cosmology, it has been usual to consider homogeneous models of the universe, filled with an idealized fluid, which at any given time has the same properties throughout the whole of its spatial extent. This procedure has a certain heuristic justification on account of the greater mathematical simplicity of homogeneous as compared with non-homogeneous models, and has a measure of observational justification on account of the approximate uniformity in the large scale distribution of extra-galactic nebulae, which is found out to the some 10^8 light-years which the Mount Wilson 100-inch telescope has been able to penetrate. Nevertheless, it is evident that some preponderating tendency for inhomogeneities to disappear with time would have to be demonstrated, before such models could be used with confidence to obtain extrapolated conclusions as to the behavior of the universe in very distant regions or over exceedingly long periods of time.

It is the object of the present note to contribute to our knowledge of the effects of inhomogeneity on the theoretical behavior of cosmological models. For the immediate purposes of this investigation we shall confine our attention to very simple models composed of dust particles (nebulae) which exert negligible pressure and which are distributed non-uniformly

but nevertheless with spherical symmetry around some particular origin. This will permit us to employ expressions for the line element and its consequences which are equivalent to those recently developed by Lemaître[1] for investigating the formation of nebulae. The result of the investigation will be to emphasize the possible dangers of drawing conclusions as to the actual universe from long range extrapolations made on the basis of a homogeneous model.

2. *The Energy-Momentum Tensor.*—For the purposes of the investigation it will be simplest to use a set of co-moving coördinates such that the spatial components are determined by a network of meshes drawn so as to connect neighboring particles and allowed to move therewith. Making use of the postulated spherical symmetry and the absence of pressure and hence also of pressure gradients, it can then readily be shown that the line element for the model can be reduced to the form

$$ds^2 = -e^{\lambda}\,dr^2 - e^{\omega}(d\theta^2 + \sin^2\theta d\phi^2) + dt^2 \qquad (1)$$

where λ and ω are functions of r and t. We may hence consider the energy-momentum tensor corresponding to our model and to this line element.

On the one hand, since the material filling the model is by hypothesis dust, exerting no pressure, we can use

$$T^{\alpha\beta} = \rho \frac{dx^{\alpha}}{ds}\frac{dx^{\beta}}{ds} \qquad (2)$$

as an expression for this tensor, where ρ is the density of the dust as measured by a local observer moving therewith, and the quantities (dx^{α}/ds) and (dx^{β}/ds) are components of the velocity of the dust with respect to the coördinates in use. And in co-moving coördinates with the line element (1), this reduces to give a single surviving component

$$T^4_4 = \rho \qquad T^{\alpha}_{\beta} = 0 \ (\alpha \text{ or } \beta \neq 4). \qquad (3)$$

On the other hand, the components of the energy-momentum tensor corresponding to the line element (1) can be computed from the formulae given for this general form by Dingle,[2] and by combining the results thus provided with the information given by (3), we easily obtain as a set of equations, connecting the metrical variables λ and ω with the density ρ,

$$8\pi T^1_1 = e^{-\omega} - e^{-\lambda}\frac{\omega'^2}{4} + \ddot{\omega} + \frac{3}{4}\dot{\omega}^2 - \Lambda = 0 \qquad (4)$$

$$8\pi T^2_2 = 8\pi T^3_3 = -e^{-\lambda}\left(\frac{\omega''}{2} + \frac{\omega'^2}{4} - \frac{\lambda'\omega'}{4}\right) + \frac{\ddot{\lambda}}{4} + \frac{\dot{\lambda}^2}{4} + \frac{\ddot{\omega}}{2} + \frac{\dot{\omega}^2}{4} + \frac{\dot{\lambda}\dot{\omega}}{4} - \Lambda = 0 \qquad (5)$$

$$8\pi T_4^4 = e^{-\omega} - e^{-\lambda}\left(\omega'' + \frac{3}{4}\omega'^2 - \frac{\lambda'\omega'}{2}\right) + \frac{\dot{\omega}^2}{4} + \frac{\dot{\lambda}\dot{\omega}}{2} - \Lambda = 8\pi\rho \qquad (6)$$

$$8\pi e^{\lambda} T_4^1 = -8\pi T_1^4 = \frac{\omega'\dot{\omega}}{2} - \frac{\dot{\lambda}\omega'}{2} + \dot{\omega}' = 0, \qquad (7)$$

where the accents denote differentiation with respect to r and the dots with respect to t, and Λ is the cosmological constant.

3. *Solution of the Equations.*—To treat these equations it is convenient to begin by eliminating λ. As a first integral of (7) we can evidently write

$$e^{\lambda} = \frac{e^{\omega}\omega'^2/4}{f^2(r)} \qquad (8)$$

where $f^2(r)$ is an undetermined function of r having values which are necessarily positive.

Substituting (8) into (1), we may then rewrite the line element in the form (3)

$$ds^2 = -\frac{e^{\omega}\omega'^2/4}{f^2(r)}dr^2 - e^{\omega}(d\theta^2 + \sin^2\theta d\phi^2) + dt^2. \qquad (9)$$

Substituting (8) into (4), we obtain

$$e^{\omega}\left(\ddot{\omega} + \frac{3}{4}\dot{\omega}^2 - \Lambda\right) + \{1 - f^2(r)\} = 0. \qquad (10)$$

As the first integral of this equation we may evidently write

$$e^{3\omega/2}\left(\frac{\dot{\omega}^2}{2} - \frac{2}{3}\Lambda\right) + 2e^{\omega/2}\{1 - f^2(r)\} = F(r), \qquad (11)$$

where $F(r)$ is a second undetermined function of r. And as the integral of this equation we can write

$$\int \frac{de^{\omega/2}}{\sqrt{f^2(r) - 1 + \frac{1}{2}F(r)e^{-\omega/2} + \frac{\Lambda}{3}e^{\omega}}} = t + \mathbf{F}(r), \qquad (12)$$

where $\mathbf{F}(r)$ is a third undetermined function.

Substituting (8) into (5), it is readily found that the result is equivalent to (10), so that further consideration of (5) is not necessary.

Finally, substituting (8) into (6), we obtain for the density of the dust

$$8\pi\rho = e^{-\omega}\left\{1 - f^2(r) - \frac{4f(r)f'(r)}{\omega'}\right\} + \frac{3}{4}\dot{\omega}^2 + \frac{\dot{\omega}\dot{\omega}'}{\omega'} - \Lambda. \qquad (13)$$

This result can be expressed in a variety of forms. Eliminating $f^2(r)$ with the help of (10), we obtain

$$8\pi\rho = -3\ddot{\omega} - 2\frac{\ddot{\omega}'}{\omega'} - \frac{3}{2}\dot{\omega}^2 - 2\frac{\dot{\omega}\dot{\omega}'}{\omega'} + 2\Lambda. \tag{14}$$

On the other hand, combining with (11), we obtain

$$8\pi\rho = \frac{e^{-3\omega/2}}{\omega'}\frac{\partial F(r)}{\partial r}. \tag{15}$$

Differentiating (15) we then obtain

$$\frac{\partial \log \rho}{\partial t} = -\frac{3}{2}\dot{\omega} - \frac{\dot{\omega}'}{\omega'} \tag{16}$$

$$\frac{\partial^2 \log \rho}{\partial t^2} = -\frac{3}{2}\ddot{\omega} - \frac{\ddot{\omega}'}{\omega'} + \left(\frac{\dot{\omega}'}{\omega'}\right)^2 \tag{17}$$

or by combining with (14) and (16)

$$\frac{\partial^2 \log \rho}{\partial t^2} = 4\pi\rho - \Lambda + \frac{1}{3}\left(\frac{\partial \log \rho}{\partial t}\right)^2 + \frac{2}{3}\left(\frac{\dot{\omega}'}{\omega'}\right)^2. \tag{18}$$

4. *Applications.*—We are now ready to consider the application of these results to the behavior of cosmological models. From a mathematical point of view, it is evident from equations (10), (11) and (12) that we can choose the three undetermined functions $f^2(r)$, $F(r)$ and $\mathbf{F}(r)$ in such a way as to correspond to any initial values of ω, $\dot{\omega}$ and $\ddot{\omega}$ as functions of r at $t = 0$ that we wish to consider, and then at least in principle could compute the later behavior of ω as a function of r and t with the help of (12). From a more physical point of view, this means that we can start our model off at $t = 0$, with $[e^{\omega}\omega'^2/4f^2(r)]$ so chosen as to give us in accordance with the form of the line element (9) any desired initial relation between the radial coördinate r and distances as actually measured from the origin, with $\dot{\omega}$ so chosen as to give us in accordance with the co-moving character of the coördinates any desired initial distribution for the measured radial velocity of the dust in the model as a function of r, with $\ddot{\omega}$ then further so chosen as to give us in accordance with (14) any desired initial distribution for the density of dust as a function of r, and with the help of our equations could then follow the later behavior of the dust composing the model. This possibility of procedure may now be applied to some specific cases.

a. Static Einstein Model.—At $t = 0$, let us choose the distributions

$$e^{\omega} = r^2 \qquad \dot{\omega} = 0 \qquad \ddot{\omega} = 0. \tag{19}$$

In accordance with (9), (10), (11) and (12) this leads to Einstein's original static cosmological line element

$$ds^2 = -\frac{dr^2}{1-\Lambda r^2} - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + dt^2, \tag{20}$$

and in accordance with (14) we should have the well-known uniform density in the Einstein model

$$4\pi\rho = \Lambda \tag{21}$$

which would remain static in agreement with (16) and (17).

b. Distorted Einstein Model.—At $t = 0$, let us choose the distributions

$$e^{\omega} = r^2 \qquad \dot{\omega} = 0 \qquad \ddot{\omega} = \ddot{\omega}_0(r), \tag{22}$$

where $\ddot{\omega}$ is initially any desired function of r.

In accordance with (14), (16) and (18) we shall then have at $t = 0$

$$4\pi\rho = \Lambda - \frac{3}{2}\ddot{\omega}_0 - \frac{1}{2}\ddot{\omega}_0' r \tag{23}$$

$$\frac{\partial \log \rho}{\partial t} = 0 \tag{24}$$

$$\frac{\partial^2 \log \rho}{\partial t^2} = 4\pi\rho - \Lambda. \tag{25}$$

Hence in this distorted Einstein model, we no longer have the uniform density of dust given by (21), and although the density of dust is initially not changing with time, the density will start to increase at those values of r where it is greater than the simple Einstein value $4\pi\rho = \Lambda$, and to decrease where it is less. This demonstrates a further kind of instability for the Einstein model—in addition to that already discussed by Eddington and others—since the initial behavior is such as to emphasize the existing differences from the uniform Einstein distribution.[4] Furthermore, in regions where the density starts to increase it is evident from the full form of equation (18) that reversal in the process of condensation would not occur short of arrival at a singular state involving infinite density or of the breakdown in our simplified equations. It will also be noted from (25) for the case of naturally flat space, $\Lambda = 0$, that *any* spherically symmetrical stationary distribution of dust would start to condense, in agreement with intuitions developed at the Newtonian level of gravitational theory.

c. Non-Static Friedmann Model.—At $t = 0$ let us choose the distributions

$$e^{\omega} = e^{g_0} r^2 \qquad \dot{\omega} = \dot{g}_0 \qquad \ddot{\omega} = \ddot{g}_0, \tag{26}$$

where g_0, $\dot{g}_0$ and $\ddot{g}_0$ are the instantaneous values corresponding to a certain function $g(t)$. In accordance with (9), (10), (11) and (12) this will lead to the known Friedmann line element for a uniform distribution of expanding or contracting dust

$$ds^2 = -e^{g(t)}\left(\frac{dr^2}{1 - r^2/R_0^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) + dt^2, \qquad (27)$$

where R_0 is a constant the value of which can be obtained with the help of (10), and $g(t)$ has the known form of dependence on t for a homogeneous model containing nothing but dust exerting negligible pressure.

In accordance with (14), (16) and (18) the initial distribution and behavior of the dust in the model would be given by

$$4\pi\rho = \Lambda - \frac{3}{2}\ddot{g}_0 - \frac{3}{4}\dot{g}_0^2 \qquad (28)$$

$$\frac{\partial \log \rho}{\partial t} = -\frac{3}{2}\dot{g}_0 \qquad (29)$$

$$\frac{\partial^2 \log \rho}{\partial t^2} = 4\pi\rho - \Lambda + \frac{1}{3}\left(\frac{\partial \log \rho}{\partial t}\right)^2. \qquad (30)$$

d. Distorted Friedmann Model.—At $t = 0$ let us choose the distributions

$$e^{\omega} = e^{g + \gamma r^2} \qquad \dot{\omega} = \dot{g}_0 \qquad \ddot{\omega} = \ddot{\omega}_0(r), \qquad (31)$$

where g_0 and $\dot{g}_0$ are the same quantities as before but $\ddot{\omega}$ is initially any desired function of r.

In accordance with (14), (16) and (18) we shall then have at $t = 0$

$$4\pi\rho = \Lambda - \frac{3}{2}\ddot{\omega}_0 - \frac{1}{2}\ddot{\omega}_0' r - \frac{3}{4}\dot{g}_0^2 \qquad (32)$$

$$\frac{\partial \log \rho}{\partial t} = -\frac{3}{2}\dot{g}_0 \qquad (33)$$

$$\frac{\partial^2 \log \rho}{\partial t^2} = 4\pi\rho - \Lambda + \frac{1}{3}\left(\frac{\partial \log \rho}{\partial t}\right)^2. \qquad (34)$$

Comparing this initial state with that for the Friedmann case, we see, although the rate of expansion or contraction in all parts of the model has been chosen the same as before, that the density of dust and the second derivative for its change with time are no longer uniform in different parts of the model. Indeed, by comparing (30) and (34) we can write for the two models at $t = 0$

$$\frac{\partial^2 \log \rho_D}{\partial t^2} - \frac{\partial^2 \log \rho_F}{\partial t^2} = 4\pi(\rho_D - \rho_F), \qquad (35)$$

where the subscripts distinguish between the Friedmann and this distorted Friedmann model. Hence at those values of r where the density in the distorted model is different from that in the Friedmann model there is at least an initial tendency for the differences to be emphasized, and from the full form of (18) it is evident in cases where condensation is taking place that the discrepancies will continue to increase until we reach a singular state involving infinite density or reach a breakdown in the simplified equations. This demonstrates a type of instability also for the Friedmann model.[4]

e. Combination of Uniform Distributions.—As a final application of the equations, we may consider an initial distribution at $t = 0$ which corresponds in a given zone, say from 0 to r_a, to the instantaneous conditions in a particular Friedmann model in accordance with the equations

$$e^{\omega} = e^{g_1 r^2} \qquad \dot{\omega} = \dot{g}_1 \qquad \ddot{\omega} = \ddot{g}_1. \tag{36}$$

This can then be surrounded by a transition zone from r_a to r_b where the values change to a new set which correspond for a further range from r_b to r_c to a different Friedmann model in accordance with the equations

$$e^{\omega} = e^{g_2 r^2} \qquad \dot{\omega} = \dot{g}_2 \qquad \ddot{\omega} = \ddot{g}_2, \tag{37}$$

and this can be followed by such additional transition zones and Friedmann zones as may be desired.

In accordance with (9), (10), (11) and (12), the dust in each Friedmann zone will then behave as in some particular completely homogeneous model without reference to the behavior of other parts of the model.[5]

5. *Conclusion.*—The foregoing results demonstrate the lack of existence of any general kind of gravitational action which would necessarily lead to the disappearance of inhomogeneities in cosmological models. This is shown both by the discovery of cases where disturbances away from an originally uniform static or non-static distribution of density would tend to increase with time, and by the possibility for models with non-interacting zones in which the behavior would agree with that for quite different homogeneous distributions.

In applying these findings to the phenomena of the actual universe, the highly simplified character of the models must of course be recognized. In the first place, although the models permit lack of homogeneity, for purposes of mathematical simplification they still retain spherical symmetry around a particular origin. Hence the phenomena of the actual universe will be affected by a more drastic kind of inhomogeneities than those here considered. In the second place, the fluid in the models was taken as dust exerting negligible pressure. Hence no allowance is made for effects such as thermal flow from one portion of matter to another

which in the actual universe might provide a *non-gravitational* kind of action which would tend to iron out inhomogeneities.

In view of the lack of complete correspondence between our models and reality we must not be too dogmatic in making assertions as to the actual universe. Nevertheless, it is at least evident from the results obtained, that we must proceed with caution in applying to the actual universe any *wide* extrapolations—either spatial or temporal—of results deduced from strictly homogeneous models. In agreement with the possibility for zones where the behavior would correspond to any desired homogeneous model, it is of course proper to treat the phenomena, in *our own neighborhood* out say to 10^8 light-years and over a *limited range of time* say 10^8 years, as approximately represented by the line element for an appropriate homogeneous model. To assert, however, that this same line element would apply to the universe as a whole, or that a homogeneous model would remain a suitable approximation at times of great condensation would not be necessarily sound. Hence, it would appear wise at the present stage of theoretical development, to envisage the possibility that regions of the universe beyond the range of our present telescopes might be contracting rather than expanding and contain matter with a density and stage of evolutionary development quite different from those with which we are familiar. It would also appear wise not to draw too definite conclusions from the behavior of homogeneous models as to a supposed initial state of the whole universe.

[1] Lemaître, *Ann. de la Soc. Scient. de Bruxelles*; **A53,** 51 (1933).

[2] These PROCEEDINGS, **19,** 559 (1933).

[3] Allowing for the difference in nomenclature, equations (9), (10), (11) and (15) are, respectively, equivalent to the equations of Lemaître, loc. cit., (8.1), (8.4), (8.2) and (8.3).

[4] Professor Dingle has been good enough to send me proof of a forthcoming article, received as this note was being completed, in which the question of the stability of homogeneous models is treated in a different manner from that here employed.

[5] This agrees with the procedure of Lemaître, loc. cit., in treating a condensing nebula surrounded by an expanding universe as analogous to two different Friedmann zones. In the present considerations we are contemplating different zones all of which are large enough to contain many nebulae.

ELASTIC WAVE ANALOGIES TO THE MOTION OF ELECTRONS IN FORCE FIELDS

BY R. B. LINDSAY

PHYSICS DEPARTMENT, BROWN UNIVERSITY

Communicated February 12, 1934

The wave mechanics method of describing the motion of an electron in a given field of force consists in setting up the Schrödinger equation using the potential energy expression V appropriate to the field and integrating subject to the assigned boundary conditions to find the ψ's, i.e., the normalized eigenfunctions. Then $\psi\bar{\psi}$ denotes the probability of finding the electron in any specified unit volume. In most cases the form of V is such that the exact solution cannot be readily obtained. However, by approximating V by a step-function considerable information can be obtained particularly for one dimensional motion,[1] such as the motion of an electron through a crystal lattice, etc. An examination of the analysis brings to light several interesting analogies between such motions and plane elastic waves in fluids, which it is the purpose of this note to point out briefly.

Consider first the one dimensional motion of an electron (total energy $W > 0$) through a rectangular potential "hollow" or "valley" of width l and depth V_0. The wave mechanical probability of transmission is

$$\left[\cos^2 \alpha_2 l + \frac{1}{4}\left(\frac{\alpha_1}{\alpha_2} + \frac{\alpha_2}{\alpha_1}\right)^2 \sin^2 \alpha_2 l\right]^{-1} \qquad (1)$$

with $\alpha_1 = \sqrt{\frac{8\pi m_0}{h^2} W} = 2\pi/\lambda_1$ and $\alpha_2 = \sqrt{\frac{8\pi m_0}{h^2} (W + V_0)} = 2\pi/\lambda_2$,

where λ_1 and λ_2 are the de Broglie wave-lengths outside and inside the hollow, respectively. On the other hand consider the passage of a compressional plane harmonic wave in a fluid through an infinite tube of cross-sectional area S_1 in which there is an abrupt constriction or expansion of length l and cross-section S_2. The ratio of the average flow of energy in the wave transmitted by the constriction or expansion to that in the wave incident on it, or the so-called *power transmission ratio*, is[2]

$$P_r = \left[\cos^2 kl + \frac{1}{4}\left(\frac{S_1}{S_2} + \frac{S_2}{S_1}\right)^2 \sin^2 kl\right]^{-1}, \qquad (2)$$

with $k = 2\pi/\lambda$, where λ is the wave-length of the compressional wave. The analogy in form between (1) and (2) is clear. Even prettier is the analogy in the case where the fluid in the expansion or contraction differs in mean density from that in the rest of the tube. Thus denoting medium

density and wave velocity by ρ and c, respectively, we have for the more general situation

$$P_r = \left[\cos^2 k_2 l + \frac{1}{4}\left(\frac{\rho_1 c_1 S_2}{\rho_2 c_2 S_1} + \frac{\rho_2 c_2 S_1}{\rho_1 c_1 S_2}\right)^2 \sin^2 k_2 l\right]^{-1}. \qquad (3)$$

It is interesting to note that if the cross-sectional areas are so chosen that $S_1/S_2 = \rho_1/\rho_2$, expression (3) reduces to

$$P_r = \left[\cos^2 k_2 l + \frac{1}{4}\left(\frac{k_2}{k_1} + \frac{k_1}{k_2}\right)^2 \sin^2 k_2 l\right]^{-1} \qquad (3')$$

which is analogous in every detail except order of magnitude of wavelengths with (1). Of course one must note the decided difference in the physical interpretation of (1) and (3'). While both are ≤ 1, the former denotes the *average* fractional number of incident electrons which get through the valley, while the latter denotes the actual wave power transmission ratio through the altered and differently confined fluid. Nevertheless, the same analysis describes both physical phenomena and the analogy can be traced at every step of the process.

As another illustration consider the motion of an electron through an unlimited one dimensional crystal lattice. Here the potential curve may to a first approximation be replaced by a succession of rectangular teeth of width l_2 separated by intervals of length l_1. The potential energy has the value V_2 over the teeth and $V_1 (< V_2)$ over the intervening spaces. If we suppose that the total energy W is greater than both V_1 and V_2, it turns out (see Frenkel, loc. cit., p. 120 ff) that the criterion for the passage of the electron through the lattice or its complete reflection is the real or imaginary nature of U, where

$$\cos U = \cos \alpha_1 l_1 \cos \alpha_2 l_2 - \frac{1}{2}\left(\frac{\alpha_1}{\alpha_2} + \frac{\alpha_2}{\alpha_1}\right) \sin \alpha_1 l_1 \sin \alpha_2 l_2, \qquad (4)$$

where now $\alpha_1 = \sqrt{\frac{8\pi m_0}{h^2}(W - V_1)}$, $\alpha_2 = \sqrt{\frac{8\pi m_0}{h^2}(W - V_2)}$. If $-1 \leqq \cos U \leq +1$, we get unit probability of transmission, otherwise total reflection ensues. There are thus certain "forbidden" energy ranges which are made readily evident by plotting $\cos U$ as a function of W. The corresponding elastic wave problem is that of plane wave transmission through an infinite one dimensional structure consisting of a series of alternating layers of two different media (either solid or fluid) where the confinement is such that there may also be an alternation in area of cross-section.[3] In this situation the transmission characteristics are given by

$$\cos U = \cos k_1 l_1 \cos k_2 l_2 - \frac{1}{2}\left(\frac{Z_2}{Z_1} + \frac{Z_1}{Z_2}\right) \sin k_1 l_1 \sin k_2 l_2 \qquad (5)$$

where l_1 and l_2 are the respective lengths of the layers $k_1 = 2\pi/\lambda_1$ and $k_2 = 2\pi/\lambda_2$, and $Z_1 = \rho_1 c_1/S_1$, $Z_2 = \rho_2 c_2/S_2$. The quantity Z is recognized as the so-called acoustic impedance of a plane compressional wave. Here again the formal agreement is evident and the analogy becomes more striking if $S_2/S_1 = \rho_2/\rho_1$. The structure here considered is of course a low-pass compressional wave filter. If cos U is plotted against frequency (instead of energy as in the electron case) cos U is numerically not greater than unity for frequencies extending from zero to some "cut-off" frequency where there begins a "forbidden" band, i.e., a range of frequencies for which no transmission takes place. The plot of (5) on a frequency scale is formally equivalent to that of (4) on an energy scale. Irreversible dissipation is, of course, ruled out in both problems.

The above analogy breaks down in the case where $W > V_1$ but $W < V_2$. In this case transmission through the lattice is controlled by

$$\cos U = \cos \alpha_1 l_1 \cosh \alpha_2 l_2 - \frac{1}{2}\left(\frac{\alpha_1}{\alpha_2} - \frac{\alpha_2}{\alpha_1}\right) \sin \alpha_1 l_1 \sinh \alpha_2 l_2 \quad (4')$$

since α_2 is now imaginary. In order to provide an elastic wave analogy it is necessary to contemplate an elastic medium in which the compressional wave velocity is infinite (i.e., there is no variation in phase from one point to another) but in which the displacement amplitude is attenuated with distance.

[1] See, for example, Frenkel, *Wave Mechanics: Elementary Theory*, Oxford, 1932, p. 99 ff.

[2] See, for example, Stewart and Lindsay, *Acoustics*, N. Y., 1930, p. 77.

[3] See Lindsay, Lewis and Albright, *Jour. Acous. Soc. Amer.*, January (1934).

A THEORY OF PROTEIN METABOLISM IN MAN

BY HENRY BORSOOK AND GEOFFREY KEIGHLEY

WILLIAM G. KERCKHOFF LABORATORIES OF THE BIOLOGICAL SCIENCES, CALIFORNIA INSTITUTE OF TECHNOLOGY

Communicated January 22, 1934

Experiments which will be published in detail elsewhere have led to the following conception of protein metabolism in man.

During a 24-hour period when the subject is maintained in nitrogen balance with either protein, or a mixture of amino acids, or a single amino acid, the larger fraction (60–75%) of the nitrogen metabolized in this period is derived not from the nitrogen ingested during this interval, but from that already present. In this sense the bulk of the protein metabo-

lized in any one day is endogenous. The term endogenous is used here in a different sense from the connotation given it by Folin, who used it to designate the wear and tear quota of protein metabolized. The simplest direct evidence for the above conclusion was furnished by experiments in which a mixture of amino acids and protein was ingested containing only a small amount of sulphur. These experiments showed that under these conditions there is no sparing of the basal sulphur. In several instances more sulphur was excreted than the sum of the amount ingested and the total excreted during a fasting period of 24 hours. It follows therefore that when protein is ingested the nitrogen and sulphur excreted do not come for the most part from the ingested protein; but that approximately three quarters is derived from protein (or protein split products) already in the organism.

The sulphur figures suggest that the ingestion of protein or amino acids is followed by a stimulation of endogenous protein metabolism. Experiments carried out to estimate the extent of this stimulation showed that it does not exceed ten per cent of the total protein metabolized, and in most cases is probably much less.

The sulphur data support the theory of the existence of depôts of "storage" protein in the organism; the amount of this storage is a function of the level of the nitrogen equilibrium. These depôts, even when the subject is in nitrogen equilibrium, are being constantly replenished and depleted. In starvation they furnish, of course, all the protein metabolized; but even when a large quantity of protein is ingested, most of the protein catabolized in a day is derived from storage protein.

The increase in uric acid following the ingestion of amino acids or proteins, which has been interpreted as evidence of a stimulation of endogenous

TABLE 1

URINARY URIC ACID

FASTING AND AFTER THE INGESTION OF AMMONIUM CARBONATE

TIME	FASTING MG.	AFTER AMMONIUM CARBONATE MG.
9 A. M.–11 A. M.	34	37
11 A. M.– 1 P. M.	41	40
1 P. M.– 3 P. M.	32	51
3 P. M.– 5 P. M.	20	45
5 P. M.– 7 P. M.	21	42
7 P. M.–11 P. M.	37	63
11 P. M.– 8 P. M.	52	54

protein metabolism,[1] is for the most part the consequence only of an increased liberation of ammonia as a result of deamination. Administration of ammonia alone is followed by an increase in urinary uric acid.

Table 1 is an abstract of the results of one of a large number of experiments in which 10–11 gm. of nitrogen in the form of ammonium carbonate was ingested. The ammonium carbonate was made up in an agar gel which was ingested over a period from 9 A. M. to 4 P. M.

Taken in conjunction with the experiments reported previously[2,3,4] the present experiments account for the salient aspects of the specific dynamic action of protein. Table 2 is a partial protocol of an experiment in which

TABLE 2

ENERGY METABOLISM, AND URINARY NITROGEN, SULPHUR AND URIC ACID FOLLOWING THE INGESTION OF 87 GM. GELATINE BETWEEN 8 A. M. AND 9 A. M.

TIME	CALORIES KG. CAL.	URINARY NITROGEN GM.	TOTAL URINARY SULPHUR MG.	TOTAL URINARY URIC ACID MG.
6.30 A. M.	62.9			
6.00–7.10				17.1
8.00				13.5
8.30	77.2			
9.00		0.60	29.6	35.8
9.30	80.4			
10.00		.83	31.8	42.7
10.30	80.5			
11.00		.98	36.9	40.9
11.30	75.5			
12.00		1.14	41.6	36.5
12.30	73.4			
1.00		0.87	38.2	20.9
1.30	70.0			
2.00		.78	45.6	20.8
2.30	70.7			
3.00		.76	51.1	23.1
3.30	69.8			
4.00		.73	49.0	21.1
4.30	70.0			
5.00		.65	49.9	13.3
5.30	70.1			
6.00				19.2

87 gm. of gelatine were ingested between 8 and 9 A. M. The results are typical of many, and it gives one of the main points bearing on the problem of the specific dynamic action of protein. The increase in uric acid closely parallels the caloric increase in metabolism. The nitrogen and sulphur excretion lag behind. We have found also that the ratio of calories in excess of the basal to nitrogen excreted in excess of the basal is not constant. It is less on a high than on a low nitrogen balance. These and other data lead us to propose the following theory regarding the specific dynamic action of protein. The increase in metabolism observed is a

composite of two groups of factors, one of which is a constant and the other variable. The constant factor comprises the increased oxygen consumption attending the oxidative deamination of the amino acids, and the conversion of the ammonia to urea. In these two processes one molecule of additional oxygen is used for every additional molecule of nitrogen deaminized and converted to urea, or approximately 7 calories per gram of nitrogen. Of the variable fraction, one part is the increase in the metabolism of the kidney while excreting the increased quantity of nitrogen. The experiments of Dock[5] have shown that the ratio of the increase in metabolism to the increase in thermodynamic work performed by the kidney diminishes as the amount of work performed increases. Under the conditions of our experiments the increased kidney work would probably not account for more than 5 calories per gram of nitrogen. The other part of the variable fraction is the stimulation of metabolism by the sudden increase in the cells, particularly of the kidney and liver, of the amount of keto acids resulting from deamination.

Under conditions where the "chemical" heat regulation mechanism of Rubner is in operation, i.e., at temperatures below 20°C., where fat and carbohydrate exert no specific dynamic action, we would predict that the specific dynamic action of protein will consist only of the constant fraction described above plus a small additional factor for the work of the kidney, i.e., between 7 and 12 kg. calories per gram of additional nitrogen metabolized. This is in excellent agreement with the values found by Rubner[6] on the dog and Terroine and Bonnet[7] on the frog. At temperatures above 25°C. where the "chemical" heat regulating mechanism is inoperative and the body temperature is governed only by the "physical" heat regulating mechanisms we should expect higher values for the specific dynamic action of protein; because here fat and carbohydrate and therefore deaminized residues now exert some stimulating influence on the metabolism. This prediction also is in full accord with the experimental findings. In dogs at 25° or higher the specific dynamic action of alanine and glycine is 14–32 kg. calories per gram of nitrogen metabolized.[8] In men after feeding gelatine we have obtained values between 18 and 32 kg. calories of excess metabolism per gram of excess nitrogen metabolized. The observation mentioned above that the specific dynamic action per gram of nitrogen metabolized is less on a high than on a low nitrogen balance can be explained reasonably on the view that a given quantity of keto acid will exert a lower stimulating effect when the fasting metabolism of deaminized residues is high than when it is low. This concept similarly affords an explanation for the observation of Wilhelmj and Mann[9] that the specific dynamic action of alanine and glycine is greater in a starving than in a well fed animal. Nevertheless, even the lowest values at the higher temperatures are higher than those obtained below 20°C., which is in accord with the theory.

Summary.—A theory of protein metabolism in man is proposed, some of whose features are:

(1) Most of the protein metabolized in any one day, even when the subject is in nitrogen equilibrium and a sufficient quantity of protein is ingested to maintain this equilibrium, is of endogenous origin, in the sense that it is derived not from the ingested protein, but from protein (or its split products) already in the body;

(2) Ammonia is one of the sources of urinary uric acid;

(3) The specific dynamic action of protein is composed of two factors, one constant, the other variable; the constant factor is the metabolism of the nitrogen; the variable factor is largely the metabolism of the deaminized residues.

[1] Rose, W. C., *J. Biol. Chem.*, **48**, 575 (1921).
[2] Borsook, H., and Winegarden, H. M., *Proc. Nat. Acad. Sci.*, **17**, 75 (1931).
[3] Borsook, H., and Keighley, G. L., *Ibid.*, **19**, 626 (1933).
[4] Borsook, H., and Keighley, G. L., *Ibid.*, **19**, 720 (1933).
[5] Dock, W., personal communication.
[6] Rubner, M., *Die Gesetze des Energieverbrauchs bei der Ernährung*, Leipzig, 1902.
[7] Terroine, E. F., and Bonnet, R., *Bull. Soc. Chim. Biol.*, **8**, 976 (1926).
[8] Wilhelmj, C. M., and Mann, F. C., *Am. J. Physiol.*, **93**, 69 (1930).
[9] Wilhelmj, C. M., and Bollman, J. L., *J. Biol. Chem.*, **77**, 127 (1928).

ON THE MOTION OF GROWTH

VIII. The Connection between Growth and Heat Production in the Amphibian, *Bufo vulgaris*, from Fertilization to Metamorphosis

BY NORMAN C. WETZEL

BABIES AND CHILDRENS HOSPITAL, AND DEPARTMENT OF PEDIATRICS, SCHOOL OF MEDICINE, WESTERN RESERVE UNIVERSITY, CLEVELAND, OHIO

Communicated February 5, 1934

Observations on growth that are limited solely to measurements of cumulative change in size are not of themselves sufficient to test the dynamical theory which applies, as we have previously shown, to human as well as to bacterial growth and metabolism.[1] Such an objective cannot, in fact, be achieved until records of size are supplemented by appropriate information in respect of energy exchange. Gayda's data[2] on the amphibian *Bufo vulgaris* also fulfil these conditions and they occupy, accordingly, a high place among similar observations on various other organisms thoroughly discussed by Needham.[3] In one respect Gayda's facts stand foremost; they possess the singular advantage of tracing the concomitant events of both growth and heat production from the moment

of fertilization to the completion of metamorphosis about 120 days later. They afford, in consequence, not merely evidence of the "initial" and "final" states of the system in question, but still more important than this, at least a hint as to what goes on during the most critical stages of reduplication. An example of this kind promises, accordingly, to constitute an unusually severe, and hence, a correspondingly significant test of theory.

Preliminary Notes.—True growth, q, is always estimated indirectly from some convenient measure of cumulative size,[4] where practicable, as weight, z, in accordance with the relation $q = \mu \log_e \frac{z}{z_0}$ and hence simply with $q = \log_e z$ if $z_0 = 1 = \mu$.[5] The corresponding data on metabolism are expressed, for a standard basal state, if possible, either directly in terms of heat (viz., as calories, $\mathfrak{T}$, calories/unit of time (Uz), or as cal./gm./unit of time, U),[4] or indirectly in terms of oxygen consumption. To an observer as well as to an analyst these measurements are basic, and they will always form the starting point for further discussions on the subject. But neither measures of size nor estimates of metabolism can legitimately be said to constitute fundamental features on the dynamical side of the problem, where stress requires to be placed, instead, upon properties hitherto described[4] as the resistance of growth, ρ, its inductance, λ, and permittance, κ. The numerical values of these constants are, however, among the last results which can be deduced in a given case, if we are required to depend, as is usual, upon data for weight and for heat alone; yet, to render the simplest account of such an undertaking, it is advantageous to begin with them.

Indeed, to bring forward good evidence that growth and concomitant basal heat production are dynamically related it is only necessary to show, since one of the constants (ρ) appears both in the equation for heat and in those for growth, that the final theoretical curves for q, z and U conform to observation with reasonable precision.

Equations for Growth and Metabolism.—For the period of 105 days beginning with the moment of fertilization and ending just short of complete metamorphosis[6] in Gayda's case of the amphibian *Bufo vulgaris*, the fundamental differential equation of growth[4] takes the form,

$$\lambda \frac{d^2q}{dt^2} + \rho \frac{dq}{dt} + \frac{q}{\kappa} = E - E_0 e^{-\beta t} \cos(\theta t + \zeta), \qquad [L^2T^{-2}\mu^{-1}] \qquad (1)$$

and represents the energy expended per unit mass and charge[5] in performing the net external work of growth, the constants λ, ρ, κ, E, E_0, β, θ and ζ, having identically the same significance and dimensions previously outlined.[4,5] This equation is similar to that found to hold true, in the human case, from early fetal to adult life, and differs from the latter only in the fact that it contains a circular rather than an hyperbolic term on the right. The complete solution of (1), giving the quantity of growth, q, at any time, t, may be written most briefly,

$$q = E\kappa - \frac{E_0 e^{-\beta t} \cos(\theta t + \zeta + \xi)}{\sqrt{\left[\lambda(\beta^2 - \theta^2) - \beta\rho + \frac{1}{\kappa}\right]^2 + [2\beta\lambda - \rho]^2\theta^2}} - C_R e^{\frac{-\rho t}{2\lambda}} \sinh(\omega t + \gamma), \ [\mu] \qquad (2)$$

wherein,

$$\xi = \tan^{-1} \frac{(2\beta\lambda - \rho)\theta}{\left[\lambda(\beta^2 - \theta^2) - \beta\rho + \frac{1}{\kappa}\right]}; \quad \omega = \frac{1}{2}\sqrt{\frac{\rho^2}{\lambda^2} - \frac{4}{\lambda\kappa}}; \quad \frac{\rho^2}{\lambda^2} > \frac{4}{\lambda\kappa},$$

C_R and γ being arbitrary for integration. With its subjoined identities this equation shows, among other things, how any finite change in growth (q) actually depends upon the fundamental constants (ρ, λ, κ, E) and hence, upon the essential properties of growth originally cited in (1). Theoretical values for q, computed from (2) as described

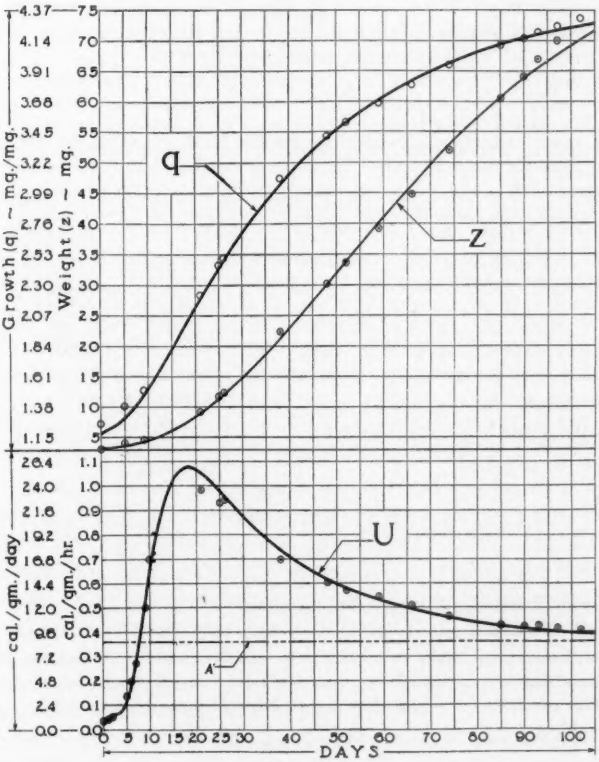


FIGURE 1

The curves of growth, cumulative gain in weight and of heat production, q, z and U, respectively, as computed from equations (2) and (3). The circles represent the mean values corrected for age in Gayda's original data. The dots centered at the twenty-second hour of the tenth day illustrate the scatter characteristic of the three individual groups of observations made during the first eleven days after fertilization.

below, and the resulting values for weight z are illustrated by the respective curves in figure 1.

Continuing with the theory outlined in this series of papers and strongly supported by the results both in the case of human and bacterial heat production, Gayda's data

on the metabolism of the toad should now conform, during the period of growth, to the equation,

$$\underbrace{\rho\left(\frac{dq}{dt}\right)^2}_{\text{Heat dissipated per unit of time and mass}} + \underbrace{\left[E_e - E_M e^{-\beta t}\sin(\theta t + \zeta')\right]\frac{dq}{dt}}_{\text{Heat liberated (or absorbed) by cellular synthesis}} + \underbrace{A'}_{\text{Heat of maintenance}} = \underbrace{U}_{\text{Resulting heat output}}, \quad [L^2T^{-3}] \quad (3)$$

in which dq/dt is understood to be given by the first derivative of (2), the several terms, as shown, possessing again the same physiological and dynamical significance as before. The constant E_M is one of a number contained in E_0 above; ζ' is not equal to ζ in (1) as it happens to be in the human case; ρ, β, θ and E_e have all appeared in (1) and (2),

TABLE 1

THE NUMERICAL VALUES OF THE CONSTANTS IN EQUATIONS (1), (2) AND (3)*

$\rho = 2404 \pm 70$ cal. day gm.$^{-1}$ μ^{-2}	$E_e = 123.9 \pm 4.1$ cal. gm.$^{-1}$ μ^{-1}
$\lambda = 13{,}900 \pm 2100$ cal. day^2 gm.$^{-1}$ μ^{-2}	$E_M = 1558 \pm 31$ cal. gm.$^{-1}$ μ^{-1}
$\kappa = 0.01740 \pm 0.00081$ gm. cal.$^{-1}$ μ^2	$\zeta' = 0.3381 \pm 0.0088$
$E = 257 \pm 13$ cal. gm.$^{-1}$ μ^{-1}	$A' = 8.646 \pm 0.050$ cal. gm.$^{-1}$ day^{-1}
$E_K = 4.478 \pm 0.077$ μ	$E_0 = 263.71$ cal. gm.$^{-1}$ μ^{-1}
$C_R = 3.74 \pm 0.14$ μ	$\beta = 0.248$ day^{-1}
$\omega = 0.0579 \pm 0.0098$ day^{-1}	$\theta = 0.1434$ day^{-1}
$\gamma = 0.824 \pm 0.048$	$\zeta = 0.39498$
$\xi = 1.52433$	

The P. E.'s of a single observation:

q, ...	± 0.017 mg./mg.
z, (Weighted for number of organisms)....	± 0.49 mg.
	± 0.35 mg.
U, ...	± 0.020 cal./gm./hr.

* While charge, or quantity of growth, q, is actually a ratio (e.g., here, mg./mg.) and is therefore a simple number, it is nevertheless advisable, to avoid confusion, that this be explicitly provided for in the dimensional formulae given above, by the symbol μ. See also reference 5.

the latter constant, however, being implied in E from the relation,[4] $E = (E_e - E_c)$. Rate of heat output per unit mass of tissue in the "resting" or no growth state (i.e., $dq/dt = 0$) is again represented by A'.

The correspondence between Gayda's original values and those obtained from (3) may be judged by the actual fit of the curve for U in figure 1, by the magnitude of the probable error of unit weight, ± 0.020 cal./gm./hr., and by the precision of the constants arranged in table 1. All adjustments are as satisfactory as we have the right to expect, for the observed values, U_D,[7] it must be recalled, are subject not only to an error in the measurement of heat itself, but to an error arising out of the observed weights, z_D, as well.

Standard methods as described by Whittaker and Robinson[8] and by Brunt[9] have been employed throughout the computations of the probable errors "of a single observation" for q, z and U likewise included in table 1, as well as for the P. E.'s of the constants themselves. Particular care has been paid to include the respective cross-

products in computing the errors of some constants necessarily determined as functions of others whose own P. E.'s had previously been calculated from the coefficients of the normal equations. Of the 12 constants in table 1 for which P. E.'s are given, the majority (10) are between 17 and 170 times as large as their corresponding measures of precision, and are thus significant. Even the most poorly determined constants are between six and seven times greater than their own P. E.'s.[10] It appears unnecessary at this time to attempt any further refinement of the tabulated values by using the latter as first approximations for a complete recomputation of the entire system. Unless further observations were at hand it would again be necessary to depend upon the data we have used, and these, it can be stated from an exhaustive study of Gayda's methods and protocols, contain an experimental error of about the same magnitude as that which represents the "probable error of a single observation" of U.

Concluding Comments.—Numerous studies on the part of antecedent investigators have established an almost universal recognition of the common outward characteristics of growth: the lagging start, the succeeding phase of acceleration, and lastly, the unavoidable decline in rate of gain as the stationary state is more nearly approached. But the corresponding trend and features of metabolism are certainly less familiar, and are accurately known, indeed, for very few examples.[3] Does rate of heat production per unit weight, to take but a single question, always pass through a maximum point, and if so, at what particular stage of development? Were this asked with the present example in mind, the methods here employed could add but a touch of numerical refinement to Gayda's original facts. Yet, to the question, *how* or even *why*, these same methods are at least likely to offer a reliable means of approach. For the results just outlined once more present good evidence to support the theory that the age changes in heat production are not only dependent upon, but are actually due to, concurrent changes in growth; in particular, that the changes which Gayda has studied succeed each other in accordance with the provisions of equations (1) to (3), and that they do so, finally, without recourse to the tenets of the surface law.

But these methods serve also as the only means, now evident, of determining the numerical value of ρ through which the events of pure growth are linked with those of heat production.

The path, which the present scheme of analysis opens, enters the heart of every problem on the subject, more specifically, wherever it is desirable to know how any one of an almost endless number of substances, factors, procedures or conditions (e.g., minerals, vitamins, hypophysectomy, temperature, etc.) actually influence the process of growth. For, if any of these really promote or inhibit growth, it is clear that they do so primarily by altering the fundamental properties represented by, and thus (it is assumed) the numerical values of, the coefficients λ, ρ and κ, in equation (1). The resulting changes in the latter will again be computed from suitable data on growth and simultaneous heat production.

These, in brief, are the principal fields of application for the methods we advocate of dealing with the general biological problem of growth. The methods themselves have again proved as reliable for the amphibian, *Bufo vulgaris*, as they have for the cognate events witnessed among unicellular populations and in man, all three examples, in fact, being represented by equations of the same general form. The latter, moreover, have been found to hold true throughout extensive periods of development, and in each instance, to represent what are commonly recognized not as one, but as two great physiological functions, growth and metabolism. Present theory, as we have seen, calls for a dynamical connection between these functions, and Gayda's facts are fully in consonance with it.

[1] Wetzel, N. C., *Proc. Soc. Exper. Biol. and Med.*, New York, **30**, 224, 227, 233, 354, 358, 360 (1932–33); *J. Pediat.*, St. Louis (in press).

[2] Gayda, T., *Arch. fisiol.*, Florence, **19**, 211–242 (1921).

[3] Needham, J., *Chemical Embryology*, London, Cambridge University Press, 1931, vol. 2.

[4] Wetzel, N. C., *Proc. Soc. Exper. Biol. and Med.*, New York, **30**, 1044–1050 (1932–33).

[5] The constant μ serves mainly to adjust the unit value of q in terms of z and z_0, the latter representing the initial body weight at $t = 0$. Thus, in the "natural" and preferable system, where $\mu = 1$, z increases exponentially with the base $e = 2.71828\ldots$ But μ may also serve to account explicitly for quantity, or charge of growth, q, in the various dimensional formulae of table 1, its omission from them being exactly analogous to the suppression of an angle in the dimensions of certain physical quantities.

[6] Metamorphosis was actually completed in some instances as early as the 112th day following fertilization, in others as late as the 131st day. Succeeding the 102nd day Gayda records but three further observations on weight and heat production, namely, for the 118th, 127th and 131st days, respectively. These were made, moreover, on only a comparatively small number of organisms (120–200), the experiments from the 20th to the 102nd day having been carried out on 300–500 subjects. Since the measurements on weight during the closing days of observation tend to be somewhat irregular, owing chiefly, no doubt, to fewer consecutive readings, but also to change in season and to the accompanying change in temperature (an increase of ca. 9°C. from April to July), it has been considered best, even though somewhat arbitrary, not to extend the present analysis beyond the 105th day following fertilization, that is, to a point just beyond the last of what appear to be the most reliable measurements. This period includes by far the greater share of the effects witnessed on growth and upon heat production, and should suffice for a rigorous test of all major premises or conclusions.

[7] The subscript "D" is used in order to distinguish the observed values for weight, growth and heat output from the corresponding theoretical values (z, q, U) without subscript.

[8] Whittaker, E. T., and Robinson, G., *The Calculus of Observations*, London, Blackie and Son, Limited, 1924.

[9] Brunt, D., *The Combination of Observations*, 2nd edition, London, Cambridge University Press, 1931.

[10] The seemingly great number of constants here involved requires a word of explanation since it is rather widely understood that any set of data, however complex, may

easily enough be represented "mathematically" if a sufficient number of terms and constants be introduced into the equations at hand. Such an objection, however, applies only to expressions known as interpolation formulae. It cannot apply to equations such as (2) in which all but two constants have merely descended from the original differential equation (1) where they represent specific and individual attributes of the system in question. The present problem amounts in some respects to this: what is the simplest differential equation representing the most general case of growth? Our own answer is given by (1) itself, provided the right-hand side is understood to be simply a special case of a more general function of t, say, $E(t)$. The resulting integrals will always contain but two arbitrary constants in addition to the constants already in (1).

THE RESOLUTION OF FOUR TESTS

BY EDWIN B. WILSON AND JANE WORCESTER

HARVARD SCHOOL OF PUBLIC HEALTH

Communicated January 31, 1934

Kelley has discussed the resolution of four tests into generals and specifics.[1] The ordinary case is that the tests may be resolved into two generals and four specifics with one degree of freedom in the resolution. As Kelley's conditions (Proposition 12, p. 52) are complicated and do not effect the resolution by formula, except implicitly, it is worth while to obtain an explicit result.

To avoid writing fractions let us adopt the notation

$$A = \frac{r_{34}}{r_{13}r_{14}}, \quad B = \frac{r_{24}}{r_{12}r_{14}}, \quad C = \frac{r_{23}}{r_{12}r_{13}} \tag{1a}$$

$$D = 1/r_{12}^2, \qquad E = 1/r_{13}^2, \qquad F = 1/r_{14}^2 \tag{1b}$$

with the understanding that the subscripts have been so applied that $A > B > C$. Then following Kelley (p. 54) in leaving α_1 arbitrary and $\beta_1 = 0$ but placing $1/\alpha_1^2 = z$, we have

$$\beta_2^2 = r_{12}^2 \frac{(B - z)(C - z)}{(A - z)} < 1 - \alpha_2^2 = r_{12}^2 (D - z) \tag{2}$$

with similar expressions for β_3^2 and β_4^2. The variable z is positive and not less than 1. As β_2^2 must be positive, z must lie between[2] A and B or between C and 1.

As z must be less than A to make β_2^2 positive, (2) may be cleared of fractions to give

$$(A + D - B - C)z < AD - BC. \tag{3}$$

If $A + D - B - C < 0$ this gives

$$z > \frac{AD - BC}{A + D - B - C} = A - \frac{(A - B)(A - C)}{A + D - B - C} \qquad (4)$$

which is greater than A. The case is therefore impossible as z must be less than A whether it be between A and B or between C and 1. Hence we may write[3]

$$1 < z < \frac{AD - BC}{A + D - B - C} = X < A. \qquad (5)$$

If we treat β_3^2 and β_4^2 in a similar manner we have, according to the range of z:

Provided $A > z > B$ Provided $C > z > 1$

$$(B + E - A - C)z > BE - AC, \quad (B + E - A - C)z < BE - AC \qquad (6)$$

$$(C + F - A - B)z > CF - AB, \quad (C + F - A - B)z < CF - AB \qquad (7)$$

$$\text{Let} \quad Y = \frac{BE - AC}{B + E - A - C}, \quad Z = \frac{CF - AB}{C + F - A - B}.$$

Conceivably there might be four cases

I. $B + E - A - C > 0$, $C + F - A - B > 0$;
II. $B + E - A - C > 0$, $C + F - A - B < 0$;
III. $B + E - A - C < 0$, $C + F - A - B > 0$;
IV. $B + E - A - C < 0$, $C + F - A - B < 0$

in which, respectively, the conditions on z imposed by (6) and (7) would be

Case	If $A > z > B$	If $C > z > 1$	Conditions[4]
I.	$X > z > Y, Z$	$X, Y, Z > z > 1$	$Y > B, Z < C$
II.	$X, Z > z > Y$	$(X, Y > z > Z)$	$Y > B, Z > C$
III.	$(X, Y > z > Z)$	$X, Z > z > Y$	$Y < B, Z < C$
IV.	$(X, Y, Z > z)$	$(X > z > Y, Z)$	$Y < B, Z > C$

Now, taking into consideration the restrictions on z as to range and upon Y and Z, it follows that the four cases in parentheses are self-contradictory and may be eliminated from further discussion.

Each of the remaining possibilities may be subdivided into a number of cases. Thus in I, if $X > Y > Z$ and $Z > 1$ we may apparently get a solution for z between A and B by taking as z any value between X and Y and we may also apparently get a solution for z by taking as z any value between Z and 1. But if Z were less than 1 the latter solution would be eliminated and if Y were less than X the former one would be cut out. The various possibilities[5] are given in the table with the evidence that the case does actually arise shown by the set of four tests specified in the last column.

TABLE OF SOLUTIONS AND TEST CASES

I_a	$A > X > z > Y > 1$ and $C > Z > z > 1$	Kelley 8, 9, 5, 1*
I_b	$A > X > z > Y > 1, Z < 1$	Kelley 2, 1, 5, 9 and 1, 3, 2, 8
I_c	$A > X > z > {}^1 > Y, Z < 1$	Kelley 7, 4, 2, 1 and 2, 3, 1, 8
I_d	$Y > C > Z > z > 1, Z < X < Y$	Kelley 1, 2, 9, 5
I_e	$Y > C > X > z > 1, Z < 1$ or $Z > X$	Kelley 4, 7, 1, 2
II_a	$A > X > z > Y > 1, Z > X$	Kelley 9, 5, 2, 1 and 8, 2, 3, 1
II_b	$A > Z > z > Y > 1, X > Z$	Kelley 1, 2, 4, 7 and *c, d, a, b*
II_c	$A > X > z > 1 > Y, Z > X$	Kelley 3, 2, 8, 1 and *b, a, d, c*
II_d	$A > Z > z > 1 > Y, X > Z$	*n, m, p, o* and *o, p, m, n*
III_a	$C > X > z > Y > 1, Z > X$	*m, n, o, p* and *p, o, n, m*
III_b	$C > Z > z > Y > 1, X > Z$	Kelley 2, 1, 7, 4 and *d, c, b, a*
III_c	$C > X > z > 1 > Y, Z > X$	*a, b, c, d*
III_d	$C > Z > z > 1 > Y, X > Z$	Kelley 5, 9, 2, 1

* The numbered tests are taken from Kelley's table of nine on p. 100 and are specified in the order which corresponds to 1, 2, 3, 4 in the type solution where $A = r_{34}/r_{13}r_{14}$, etc., with $A > B > C$; the coefficients corrected for attenuation were used. The sets of tests *a, b, c, d* and *m, n, o, p* are defined by the coefficients $r_{ab} = \sqrt{5}/3$, $r_{ac} = 2/\sqrt{11}$, $r_{ad} = 1/\sqrt{11}$, $r_{bc} = 4\sqrt{5}/(3\sqrt{11})$, $r_{bd} = 5/\sqrt{11}$, $r_{cd} = 8/11$, $r_{mn} = 6/\sqrt{69}$, $r_{mo} = 2/3$, $r_{mp} = 1/\sqrt{7}$, $r_{no} = 8/\sqrt{69}$, $r_{np} = 15/\sqrt{483}$, $r_{op} = 2/\sqrt{7}$. One may note that the same set of four tests but in different orders may illustrate more than one of the possible types.

Kelley raises the question (Proposition 14, p. 56) as to whether if four tests are such that taken three at a time they are resoluble into one general and three specifics, they must be resoluble into two generals and four specifics. The answer is: No—as shown by the following case[7]

$$r_{12} = \frac{1}{\sqrt{14}},\ r_{13} = \frac{1}{4},\ r_{14} = \frac{\sqrt{5}}{11},\ r_{34} = \frac{13\sqrt{5}}{44},\ r_{24} = \frac{9\sqrt{5}}{11\sqrt{14}},\ r_{23} = \frac{5}{4\sqrt{14}}.$$

The communalities, as Thurstone[6] calls them, are $h_1^2 = \alpha_1^2 + \beta_1^2$, $h_2^2 = \alpha_2^2 + \beta_2^2$, $h_3^2 = \alpha_3^2 + \beta_3^2$, $h_4^2 = \alpha_4^2 + \beta_4^2$. Now if $z = 1$, $h_1 = 1$; if $z = X$, $h_2 = 1$; if $z = Y$, $h_3 = 1$; if $z = Z$, $h_4 = 1$. When a communality is unity there is nothing left over for the specific. Looking over the 13 solutions in the table it is seen that for each range of z the limits are two of the four quantities 1, X, Y, Z and hence one may infer that if a set of four tests may be resolved into two generals and four specifics, it may be resolved in such a way that either of two of the specifics shall vanish and in the case I_a which has a double range the resolution may be accomplished in such a manner that any one of the four specifics shall vanish. If for a set of four tests the raw coefficients of correlation were used any communality in excess of the reliability coefficient would presumably be of doubtful psychological significance.

[1] Kelley, T. L., *Crossroads in the Mind of Man*, pp. 46–57.

[2] To avoid complications we have assumed that r_{12}, r_{13}, r_{14} are none of them zero and we shall use inequalities even though in extreme positions the inequality might become

an equality. As the variables are continuous such extremes and the cases where r_{12}, r_{13}, r_{14} were zero could be treated as limits.

[3] This means that the largest (A) of the ratios in (1a) must be greater than (or equal to) unity and is therefore equivalent to Hotelling's necessary condition (p. 55) as we could work with any of the four rows in [37].

[4] These restrictions on Y and Z come from expressions like that in (4) for X.

[5] If z is known, the complete solution of the resolution is at hand as $\alpha_1 = r_{ag} = 1/\sqrt{z}$, $\alpha_2 = r_{bg} = \sqrt{z}r_{12}$, $\alpha_3 = r_{cg} = \sqrt{z}r_{13}$, $\alpha_4 = r_{dg} = \sqrt{z}r_{14}$, $\beta_1 = r_{ah} = 0$, $\beta_2 = r_{bh} = Kr_{12}/(A - z)$, $\beta_3 = r_{ch} = Kr_{13}/(\beta - z)$, $\beta_4 = r_{dh} = Kr_{14}/(C - z)$. Where **a, b, c, d** are the tests, **g** and **h** are the two generals, and $K = \sqrt{(A - z)(B - z)(C - z)}$. The parts of **g** and **h** which are determined, i.e., which lie in the space of **a, b, c, d** and which give the regression values of **g** and **h** may be written at once as $\mathbf{g}_0 = r_{ag}\mathbf{a}' + r_{bg}\mathbf{b}' + r_{cg}\mathbf{c}' + r_{dg}\mathbf{d}'$, $\mathbf{h}_0 = r_{ah}\mathbf{a}' + r_{bh}\mathbf{b}' + r_{ch}\mathbf{c}' + r_{dh}\mathbf{d}'$, where $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$, $\mathbf{d}'$ are the set reciprocal to **a, b, c, d.** (If one wishes unit team tests in place of $\mathbf{g}_0$ and $\mathbf{h}_0$, one need only divide $\mathbf{g}_0$ and $\mathbf{h}_0$ by their magnitudes.) We may write $\mathbf{g} = \mathbf{g}_0 + \mathbf{v}$, $\mathbf{h} = \mathbf{h}_0 + \mathbf{w}$, if **v** and **w** are suitable vectors in the $n - 5$ space completely perpendicular to a, b, c, d and the line equally inclined to the n axes, as in E. B. Wilson, these PROCEEDINGS, **14,** p. 286 (1930).

[6] Thurstone, L. L., *Psychological Rev.*, **38,** 406–427 (1931) and *The Theory of Multiple Factors*, The University of Chicago Bookstore.

[7] If each set of three tests is resoluble into one general and three specifics all the correlation coefficients may be taken positive and all the first partials must then be positive. In terms of A, B, C, D, E, F this means that A, B, C are greater than 1, that $D > B$, C and $E > A$, C and $F > A$, B, that $AD - BC > 0$, $BE - AC > 0$, $CF - AB > 0$. Now if $F = A + \alpha$, $E = A + \beta$, $D = B + \gamma$

$$A + D - B - C = A - C + \gamma \qquad B + E - A - C = B - C + \beta,$$

$$C + F - A - B = C - B + \alpha.$$

These will all be positive with the possible exception of the third. Also

$$AD - BC = A(B + \gamma) - BC, \qquad BE - AC = B(A + \beta) - AC,$$

$$CF - AB = C(A + \alpha) - AB.$$

If we were under case I we should have $\alpha > (B - C)$ and all three of these expressions would be positive. If then we should have $Z < 1$ and $X < Y$ there could be no solution. Then $(C - 1)\alpha < (B - C)(A - 1)$ and $\beta\gamma < (A - C)(A - B)$. If therefore we can find α between $B - C$ and $B - C$ multiplied by $(A - 1)/(C - 1)$ and $\beta\gamma$ less than $(A - C)(A - B)$ and can so choose A, B, C that the correlation coefficients are a consistent set, we shall have an example for which there is no solution. We chose $A = 13$, $B = 9$, $C = 5$, $\alpha = 11.2$, $\beta = 3$, $\gamma = 5$ in the example given. Such an analysis would obviously find other examples; a similar discussion was used to find the resoluble groups (a, b, c, d), (m, n, o, p) needed to show the existence of certain types of solution as tabulated for which we had not happened across any cases from among Kelley's nine tests.

ON RESOLUTION INTO GENERALS AND SPECIFICS

BY EDWIN B. WILSON

HARVARD SCHOOL OF PUBLIC HEALTH

Communicated February 14, 1934

If a set of tests $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots k$ in number may be considered as resoluble[1] into a certain number of generals and k specifics so that

$$\mathbf{a} = \mathbf{a}.\mathbf{g}\mathbf{g} + \mathbf{a}.\mathbf{h}\mathbf{h} + \ldots + \sqrt{1 - (\mathbf{a}.\mathbf{g})^2 - (\mathbf{a}.\mathbf{h})^2 - \ldots}\ \mathbf{s}_a, \quad (1)$$

$$\mathbf{b} = \mathbf{b}.\mathbf{g}\mathbf{g} + \mathbf{b}.\mathbf{h}\mathbf{h} + \ldots + \sqrt{1 - (\mathbf{b}.\mathbf{g})^2 - (\mathbf{b}.\mathbf{h})^2 - \ldots}\ \mathbf{s}_b, \quad (2)$$

etc., the generals $\mathbf{g}, \mathbf{h}, \ldots$ and the specifics $\mathbf{s}_a, \mathbf{s}_b, \ldots$, being by definition all mutually orthogonal, must reach out into a space of dimensionality higher than that of the k-dimensional space of the k vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$. It is hopeless to determine the generals and specifics; but one may hope to determine their projections on the k-space of the tests. As (1), (2), ... must hold both in respect to their projections on any space and in respect to their components perpendicular to that space, let $\mathbf{g}_0, \mathbf{h}_0, \ldots \mathbf{s}_{a0}, \ldots$ be the projections of $\mathbf{g}, \mathbf{h}, \ldots \mathbf{s}_a, \mathbf{s}_b \ldots$ on the space of the tests; then, as $\mathbf{a}.\mathbf{g}_0, \mathbf{a}.\mathbf{h}_0, \ldots, \mathbf{b}.\mathbf{g}_0, \ldots$ must equal $\mathbf{a}.\mathbf{g}, \mathbf{a}.\mathbf{h}, \mathbf{b}.\mathbf{g}, \ldots$ respectively, we may write in terms of vectors entirely within the space of operation,

$$\mathbf{a} = \mathbf{a}.\mathbf{g}_0\mathbf{g}_0 + \mathbf{a}.\mathbf{h}_0\mathbf{h}_0 + \ldots + \sqrt{1 - (\mathbf{a}.\mathbf{g}_0)^2 - (\mathbf{a}.\mathbf{h}_0)^2 - \ldots}\ \mathbf{s}_{a0} \quad (1')$$

$$\mathbf{b} = \mathbf{b}.\mathbf{g}_0\mathbf{g}_0 + \mathbf{b}.\mathbf{h}_0\mathbf{h}_0 + \ldots + \sqrt{1 - (\mathbf{b}.\mathbf{g}_0)^2 - (\mathbf{b}.\mathbf{h}_0)^2 - \ldots}\ \mathbf{s}_{b0} \quad (2')$$

etc. As $\mathbf{s}_a$ is perpendicular to $\mathbf{g}, \mathbf{h}, \ldots, \mathbf{s}_b, \mathbf{s}_c, \ldots$ it must be perpendicular to $\mathbf{b}, \mathbf{c}, \ldots$ which are linearly compounded of them and since the component of $\mathbf{s}_a$ perpendicular to the k-space of the tests is perpendicular to the tests, it follows at once that the projection $\mathbf{s}_{a0}$ must be perpendicular to $\mathbf{b}, \mathbf{c}, \ldots$, the projection $\mathbf{s}_{b0}$ must be perpendicular to $\mathbf{a}, \mathbf{c}, \ldots$, etc. But the vectors $\mathbf{a}', \mathbf{b}', \mathbf{c}', \ldots$ of the reciprocal set are respectively perpendicular to all the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \ldots$ except $\mathbf{a}, \mathbf{b}, \mathbf{c} \ldots$ individually. Hence $\mathbf{s}_{a0}, \mathbf{s}_{b0}, \ldots$ are parallel to $\mathbf{a}', \mathbf{b}', \ldots$, respectively, and we may write

$$\mathbf{a} = \mathbf{a}.\mathbf{g}_0\mathbf{g}_0 + \mathbf{a}.\mathbf{h}_0\mathbf{h}_0 + \ldots x\mathbf{a}',$$
$$\mathbf{a}.\mathbf{a} = 1 = (\mathbf{a}.\mathbf{g}_0)^2 + (\mathbf{a}.\mathbf{h}_0)^2 + \ldots + x,$$
$$\mathbf{a} = \mathbf{a}.\mathbf{g}_0\mathbf{g}_0 + \mathbf{a}.\mathbf{h}_0\mathbf{h}_0 + \ldots + [1 - (\mathbf{a}.\mathbf{g}_0)^2 - (\mathbf{a}.\mathbf{h}_0)^2 - \ldots]\mathbf{a}' \quad (1'')$$

The expression $(\mathbf{a}.\mathbf{g}_0)^2 + (\mathbf{a}.\mathbf{h}_0)^2 + \ldots$ is the communality h_a^2 in Thurstone's terminology.

Until we know the values of the communalities $h_a^2, h_b^2, \ldots$ we cannot tell how large a component of $\mathbf{a}, \mathbf{b}, \ldots$ lies along $\mathbf{a}', \mathbf{b}', \ldots$; but the form of the equation (1″) shows that the team-test[2] for $\mathbf{s}_a$, i.e., the unit vector in the space of the tests which correlates most highly with $\mathbf{s}_a$ (because it

lies in the direction of the projection of $\mathbf{s}_a$ upon that space) is the unit vector along $\mathbf{a}'$. *We therefore know the team-tests for all the specifics whether we have resolved the tests into generals or specifics or not, and even before we know how many generals are to be involved in the resolution.* Indeed if I be the "idemfactor,"[3]

$$\mathbf{a} = \mathbf{a}.\mathrm{I} = \mathbf{a}.(\mathbf{a}\mathbf{a}' + \mathbf{b}\mathbf{b}' + \mathbf{c}\mathbf{c}' + \ldots) = \mathbf{a}' + r_{ab}\mathbf{b}' + r_{ac}\mathbf{c}' + \ldots \quad (3)$$

$$\mathbf{b} = r_{ab}\mathbf{a}' + \mathbf{b}' + r_{bc}\mathbf{c}' + \ldots, \quad \mathbf{c} = r_{ac}\mathbf{a}' + r_{bc}\mathbf{b}' + \mathbf{c}' + \ldots \quad (3')$$

Now if R_{aa}, R_{bb}, ... be the cofactors of the determinant

$$D = \begin{vmatrix} 1 & r_{ab} & r_{ac} & \ldots \\ r_{ab} & 1 & r_{bc} & \ldots \\ r_{ac} & r_{bc} & 1 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} \quad (4)$$

equations (3), (3′) may be solved at once for the reciprocals as

$$\mathbf{a}' = (R_{aa}\,\mathbf{a} + R_{ab}\,\mathbf{b} + R_{ac}\,\mathbf{c} + \ldots)/D, \text{ etc.}, \quad (5)$$

and one may show by actual multiplication and reduction that

$$\mathbf{a}'.\mathbf{a}' = R_{aa}/D, \ \mathbf{a}'.\mathbf{b}' = R_{ab}/D, \text{ etc.} \quad (6)$$

Hence the team-tests for the specifics are

$$\mathbf{t}_a = \frac{\mathbf{a}'}{\sqrt{\mathbf{a}'.\mathbf{a}'}} = \frac{1}{\sqrt{R_{aa}D}}\,(R_{aa}\,\mathbf{a} + R_{ab}\,\mathbf{b} + R_{ac}\,\mathbf{c} + \ldots) \quad (7)$$

$$\mathbf{t}_b = \frac{\mathbf{b}'}{\sqrt{\mathbf{b}'.\mathbf{b}'}} = \frac{1}{\sqrt{R_{bb}D}}\,(R_{ab}\,\mathbf{a} + R_{bb}\,\mathbf{b} + R_{bc}\,\mathbf{c} + \ldots) \quad (7')$$

These teams do not represent the regression values or known parts of the specifics because $\mathbf{s}_{a0}$, $\mathbf{s}_{b0}$ are not unit vectors. Also the teams $\mathbf{t}_a$, $\mathbf{t}_b$, ... are correlated, and indeed with the same correlation coefficient as $\mathbf{s}_{a0}$, $\mathbf{s}_{b0}$, ..., namely,

$$\mathbf{t}_a.\mathbf{t}_b = \mathbf{a}'.\mathbf{b}'/\sqrt{\mathbf{a}'.\mathbf{a}'\mathbf{b}'.\mathbf{b}'} = R_{ab}/\sqrt{R_{aa}R_{bb}}, \text{ etc.} \quad (8)$$

Further, the form of (7), (7′), ... shows that the team-tests for each specific depend on all the tests. And finally from the fact that the set reciprocal to $\mathbf{a}', \mathbf{b}', \mathbf{c}', \ldots$ is $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ we may infer that *if in the resolution of* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ *into generals and specifics, the team-tests for the specifics be* $\mathbf{t}_a, \mathbf{t}_b, \mathbf{t}_c, \ldots$, *then if psychological tests should be set up for these specifics, the resolution of these tests into generals and specifics would have* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ *as the team-tests for the (determinable parts of) the specifics.*[4]

If the communalities [5] h_a^2, h_b^2, ... were known the vectors

$$\mathbf{a} - (1 - h_a^2)\mathbf{a}', \ \mathbf{b} - (1 - h_b^2)\mathbf{b}', \ \mathbf{c} - (1 - h_c^2)\mathbf{c}', \ldots, \quad (9)$$

being linearly compounded of the (determinable parts of) the generals $\mathbf{g}_0$, $\mathbf{h}_0$, ... as in (1″), must lie in a space within the k-space of the tests of a number of dimensions equal to the number of generals used in the resolution. From (3), (3′) we may express the vectors (8) as components along $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$, ... in the tabular form

$$\begin{array}{llllll} \text{component along} & & \mathbf{a}' & \mathbf{b}' & \mathbf{c}' & \dots \\ \text{of } \mathbf{a} - (1 - h_a^2)\mathbf{a}' & \text{is} & h_a^2 & r_{ab} & r_{ac} & \dots \\ \text{of } \mathbf{b} - (1 - h_b^2)\mathbf{b}' & \text{is} & r_{ab} & h_b^2 & r_{bc} & \dots \\ \text{of } \mathbf{c} - (1 - h_c^2)\mathbf{c}' & \text{is} & r_{ac} & r_{bc} & h_c^2 & \dots \\ \dots\dots\dots\dots & & \dots\dots\dots\dots & & & \end{array} \tag{10}$$

Hence the rank of the matrix in (10) made up of the coefficients of the reciprocal set is equal to the number of generals used in the resolution. This is perhaps as neat a proof as any of Thurstone's fundamental theorem. Another proof would be found in noting that we have $k(k-1)/2$ equations of the form

$$\mathbf{a}.\mathbf{b} = \mathbf{a}.\mathrm{I}.\mathbf{b} = \mathbf{a}.\mathbf{g}_0\mathbf{g}_0.\mathbf{b} + \mathbf{a}.\mathbf{h}_0\mathbf{h}_0.\mathbf{b} + \dots \tag{11}$$

$$\mathbf{a}.(\mathrm{I} - \mathbf{g}_0\mathbf{g}_0 - \mathbf{h}_0\mathbf{h}_0, \dots).\mathbf{b} = \mathbf{a}.\Phi.\mathbf{b} = 0. \tag{11'}$$

As $\mathbf{a}.\Phi$ is therefore perpendicular to $\mathbf{b}$, $\mathbf{c}$, ..., it must be parallel to $\mathbf{a}'$ and indeed from (1″) must be equal to $(1 - h_a^2)\mathbf{a}'$. Hence

$$\Phi = \mathrm{I} - \mathbf{g}_0\mathbf{g}_0 - \mathbf{h}_0\mathbf{h}_0 - \dots = (1 - h_a^2)\mathbf{a}'\mathbf{a}' + (1 - h_b^2)\mathbf{b}'\mathbf{b}' + \dots \tag{12}$$

$$\Psi = \mathbf{g}_0\mathbf{g}_0 + \mathbf{h}_0\mathbf{h}_0 + \dots = [\mathbf{a} - (1 - h_a^2)\mathbf{a}']\mathbf{a}' + [\mathbf{b} - (1 - h_b^2)\mathbf{b}']\,\mathbf{b}' + \dots$$

Again expressing $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$, by (2), (2′) we have for Ψ

$$\begin{aligned} \Psi = \mathbf{g}_0\mathbf{g}_0 + \mathbf{h}_0\mathbf{h}_0 + \dots = {} & h_a^2\mathbf{a}'\mathbf{a}' + r_{ab}\mathbf{a}'\mathbf{b}' + r_{ac}\mathbf{a}'\mathbf{c}' + \dots \\ & + r_{ab}\mathbf{b}'\mathbf{a}' + h_b^2\mathbf{b}'\mathbf{b}' + r_{bc}\mathbf{b}'\mathbf{c}' + \dots \\ & + r_{ac}\mathbf{c}'\mathbf{a}' + r_{bc}\mathbf{c}'\mathbf{b}' + h_c^2\mathbf{c}'\mathbf{c}' + \dots \\ & + \dots\dots\dots\dots\dots\dots\dots\dots \end{aligned} \tag{12'}$$

and the matrix of the coefficients must be of rank equal to the number of generals, because Ψ has that rank, as is obvious from its form.

It has been shown[1c] that the number of conditions on k tests to effect a resolution into l generals and k specifics is $T = (k-1)^2/2 - (k+1)/2$; if T should turn out to be negative its value would determine the number of degrees of freedom in a resolution if it were possible. The necessity for specifying the possibility inheres in the fact that, because of the quadratic nature of the equations involved, the resolution, to be real, may require the use of more generals than the minimum necessity to make T zero or negative.[6] It is also well known that, when there is more than one general, even the determinable parts of the generals are not unique because an orthogonal transformation (involving $l(l-1)/2$ constants)

may be applied to the generals. The parts $\mathbf{g}_0$, $\mathbf{h}_0$, ... of the generals are not mutually orthogonal but the transformation upon them has the same coefficients as an orthogonal transformation. Indeed the vectors $\mathbf{g}_0$, $\mathbf{h}_0$, ... under the whole group of transformations trace the surface of the hyper-ellipsoid which is the projection upon the space of the tests of the hyper-sphere traced by the generals $\mathbf{g}$, $\mathbf{h}$, ... under the orthogonal transformations. This ellipsoid will have a number of axes equal to the number of the generals used and none of the axes may be greater than unity. The axes (except in case two or more of them should be equal) will be determinate, and if that particular set of the vectors $\mathbf{g}_0$, $\mathbf{h}_0$, ... which lie along those axes be taken, the vectors will be determined except for sign. If, however, $T < 0$, there will remain an indeterminacy of the ellipsoid itself leading to a family of ∞^T such ellipsoids. We thus see that *for any resolution into generals and specifics the determinable parts* $\mathbf{g}_0$, $\mathbf{h}_0$, ... *of the generals may be taken as mutually perpendicular (uncorrelated) and if so taken the resolution is unique except for sign, except for interchange between* $\mathbf{g}_0$, $\mathbf{h}_0$, ..., *and except as* T *be negative.* Actually to effect the resolution would be a long arithmetic procedure.[6b,c]

[1] The previous notes on this general field are found in: (*a*) *Science*, **67**, 244–248 (1928); (*b*) these PROCEEDINGS, **14**, 283–291 (1928); (*c*) *J. Gen. Psychol.*, **2**, 153–169 (1929); (*d*) *J. Educat. Psychol.*, March 1929, 217–223; (*e*) these PROCEEDINGS, **19**, 768–772 (1933); (*f*) *Ibid.*, 882–884; (*g*) *Ibid.*, 1039–1044; (*h*) *Ibid.*, **20**, 189–192 (1934).

[2] The term "team" is here used for a specific as Spearman uses it for a general.

[3] For the dyadic notation see E. B. Wilson, *Vector Analysis*, Chaps. V and VI generalized to take care of operations in more than three dimensions.

[4] The reciprocity is worth pondering for light it may throw on the significance of resolution into generals and specifics; apparently any test may be regarded as the team for a specific, but not for the specific of that test!

[5] L. L. Thurstone's term; see (*a*) *Psychol. Rev.*, **38**, 406–427 (1931); (*b*) *Theory of Multiple Factors*, 1933, Chicago University Bookstore; (*c*) *Simplified Multiple Factor Method*, 1933, Chicago Univ. Bookstore.

[6] The form of the values of l and k which will make $T = 0$ is $l = \frac{1}{2}p(p+1)$, $k = \frac{1}{2}(p+1)(p+2)$, $p = 1, 2, \ldots$; so that l runs through the values 1, 3, 6, 10, 15, ... and k through 3, 6, 10, 15, 21, ... The following table gives the value of T for different values of k and l:

	$k =$	3	4	5	6	7	8	9	10	...
$l = 1$,	$T =$	0	2	5	9	14	20	27	35	...
$l = 2$,		−2	−1	1	4	8	13	19	26	...
$l = 3$,		−3	−3	−2	0	3	7	12	18	...
$l = 4$,			−4	−4	−3	−1	2	6	11	...
$l = 5$,				−5	−5	−4	−2	1	5	...
$l = 6$,					−6	−6	−5	−3	0	...

BOOLEAN ALGEBRAS AND THEIR APPLICATION TO TOPOLOGY[1]

BY M. H. STONE

DIVISION OF MATHEMATICS, HARVARD UNIVERSITY

Read before the Academy, Wednesday, November 22, 1933

The purpose of this note is to present the essential features of two papers now in preparation for submission to the Transactions of the American Mathematical Society.

A Boolean algebra may be defined by a set of postulates in terms of (logical) addition and (logical) multiplication as undefined operations.[2] It is convenient, in algebraic discussions at least, to admit the customarily excluded case in which the algebra contains just one element. By analogy with the theory of abstract rings,[3] we are led to consider a theory of ideals and homomorphisms for Boolean algebras. An ideal $\mathfrak{a}$ in a Boolean algebra A is a class of elements belonging to A with the properties:

(1) if a and b are in $\mathfrak{a}$, their sum $a + b$ is in $\mathfrak{a}$;
(2) if a is in $\mathfrak{a}$ and b is in A, their product ab is in $\mathfrak{a}$.

Two elements b, c in A are said to be congruent (mod $\mathfrak{a}$) if their symmetric difference[4] is an element in the ideal $\mathfrak{a}$. An ideal $\mathfrak{a}$ determines a partition of the elements belonging to A into disjoint subclasses $\mathfrak{k}$ of elements mutually congruent (mod $\mathfrak{a}$) and does so in such a manner that the subclasses $\mathfrak{k}$ constitute a Boolean algebra A^* when the sum (product) of $\mathfrak{k}_1$ and $\mathfrak{k}_2$ is defined as the class $\mathfrak{k}_3$ which contains the sum (product) of some element in $\mathfrak{k}_1$ and some element in $\mathfrak{k}_2$; this algebra A^* is called the quotient of A by $\mathfrak{a}$ and is denoted by $A/\mathfrak{a}$. The fundamental theorem concerning homomorphisms may now be stated as follows:

THEOREM I. *A necessary and sufficient condition that a Boolean algebra B be a homomorph of the Boolean algebra A, is that there exist an ideal $\mathfrak{a}$ in A such that B is an isomorph of $A/\mathfrak{a}$. If B is a given homomorph of A, the ideal $\mathfrak{a}$ is uniquely determined as the class of all those elements belonging to A which are antecedents, in the homomorphism, of the zero element in B.*

An ideal $\mathfrak{a}$ is said to be prime if it is a proper subclass of A contained in no ideal other than $\mathfrak{a}$ itself and the unit ideal $\mathfrak{e}$ comprising all the elements in A. An ideal $\mathfrak{a}$ is prime if and only if $A/\mathfrak{a}$ is an algebra of exactly two elements. By relying, in the general case, upon the well-ordering hypothesis, we obtain the following theorem.[5]

THEOREM II. *If A is a Boolean algebra containing two unequal elements, a and b, then there exists a prime ideal $\mathfrak{a}$ in A which contains one, but not both, of the elements a and b.*

From this theorem we can deduce the following result:

THEOREM III. *Let A be an arbitrary Boolean algebra, $\mathfrak{E}$ the set of all prime ideals in A, $\mathfrak{E}(a)$ the set of all prime ideals in A which do not contain the element a, and $B(A)$ the algebra of the sets $\mathfrak{E}(a)$ with sum and product defined as union and intersection, respectively. Then the correspondence $a \longleftrightarrow \mathfrak{E}(a)$ defines an isomorphism of A and $B(A)$.*[6]

We shall now connect the theory of Boolean algebras with abstract topology by means of three fundamental theorems. We shall use the word "space" to signify "Hausdorff space," and shall refer the reader to the literature for the standard nomenclature and notation.[7]

THEOREM IV$_1$. *Let A, $\mathfrak{E}$, and $\mathfrak{E}(a)$ be defined as in Theorem III, and let a topology be introduced in $\mathfrak{E}$ by the assignment of each set $\mathfrak{E}(a)$ as a neighborhood of every element which it contains. Then the set $\mathfrak{E}$ is converted into a space $\mathfrak{S}(A)$ with the following properties:* (1) *$\mathfrak{S}(A)$ is bicompact;* (2) *$\mathfrak{S}(A)$ is totally disconnected*[8]; (3) *a subset of $\mathfrak{S}(A)$ is both open and closed if and only if it is a set $\mathfrak{E}(a)$. In order that two Boolean algebras, A_1 and A_2, be isomorphic, it is necessary and sufficient that $\mathfrak{S}(A_1)$ and $\mathfrak{S}(A_2)$ be equivalent.*[9]

THEOREM IV$_2$. *Let $\mathfrak{S}$ be a totally-disconnected bicompact space, and let $A[\mathfrak{S}]$ be the algebra of all those subsets of $\mathfrak{S}$ which are both open and closed. Then $A[\mathfrak{S}]$ is a Boolean algebra, and the corresponding space $\mathfrak{S}(A[\mathfrak{S}])$ defined in Theorem IV$_1$ is equivalent to $\mathfrak{S}$.*

THEOREM IV$_3$. *Let A, $\mathfrak{E}(a)$, and $\mathfrak{S}(A)$ be defined as in Theorems III and IV$_1$. Then there exists a one-to-one correspondence, $\mathfrak{a} \longleftrightarrow \mathfrak{G}(\mathfrak{a})$, between the class of all ideals $\mathfrak{a}$ in A and the class of all open sets $\mathfrak{G}$ in $\mathfrak{S}(A)$ such that an element a is in $\mathfrak{a}$ if and only if $\mathfrak{E}(a)$ is contained in $\mathfrak{G}(\mathfrak{a})$. By virtue of this correspondence, the space $\mathfrak{S}(A/\mathfrak{a})$ is equivalent to the complement of $\mathfrak{G}(\mathfrak{a})$ in $\mathfrak{S}(A)$.*

These theorems show that the theory of Boolean algebras is coextensive with the theory of totally-disconnected bicompact spaces. In the sequel, therefore, we shall refer to spaces of this type as Boolean spaces. By a simple modification of a theorem due to Tychonoff,[10] we find the following result:

THEOREM IV$_4$. *Every Boolean space of character $\mathfrak{c}$ is equivalent to a subset of a universal Boolean space $\mathfrak{B}(\mathfrak{c})$ which has the character $\mathfrak{c}$. In consequence, every Boolean algebra of cardinal number $\mathfrak{c}$ is a homomorph of a universal Boolean algebra $A(\mathfrak{c})$ which has the cardinal number $\mathfrak{c}$; and every Boolean algebra of cardinal number not greater than $\mathfrak{c}$ is a homomorph of a subalgebra of $A(\mathfrak{c})$.*

We shall now apply the foregoing theory to the study of arbitrary spaces.[11] It will be convenient to exclude from consideration algebras with only one element and spaces which are void. Let $\mathfrak{S}$ be an arbitrary normal space and α a class of mutually disjoint non-void closed subsets $\mathfrak{A}$ of $\mathfrak{S}$ with union $\mathfrak{S}_\alpha$. We may take the sets $\mathfrak{A}$ as the points of a space $\mathfrak{R}(\mathfrak{S},\alpha)$

if we define neighborhoods as follows: if $\mathfrak{A}_0$ is an arbitrary set in α and $\mathfrak{G}$ is an arbitrary open subset of $\mathfrak{S}$ containing $\mathfrak{A}_0$, then the class $\mathfrak{U}(\mathfrak{A}_0,\mathfrak{G})$ of all sets $\mathfrak{A}$ contained in $\mathfrak{G}$ is a neighborhood of $\mathfrak{A}_0$. If a space $\mathfrak{R}$ is equivalent to the space $\mathfrak{R}(\mathfrak{S},\alpha)$ so defined, the relation between $\mathfrak{R}$, $\mathfrak{S}$, and α is said to be a map, $m(\mathfrak{R}, \mathfrak{S}, \alpha)$, of $\mathfrak{R}$ in $\mathfrak{S}$ determined by α. It is obvious that, if $\mathfrak{R}_2,\mathfrak{S}_2$ are equivalent to $\mathfrak{R}_1,\mathfrak{S}_1$, respectively, and if α_2 is the class corresponding to α_1 by the equivalence of $\mathfrak{S}_2$ and $\mathfrak{S}_1$, then the existence of a map $m(\mathfrak{R}_1,\mathfrak{S}_1,\alpha_1)$ implies and is implied by the existence of a map $m(\mathfrak{R}_2,\mathfrak{S}_2,\alpha_2)$; two maps which are so related are called equivalent. The class α is said to cover $\mathfrak{S}$ if $\mathfrak{S}_\alpha = \mathfrak{S}$; is said to be densely distributed in $\mathfrak{S}$ if every non-void open subset of $\mathfrak{S}$ contains a set $\mathfrak{A}$ in α; and is said to be continuous[12] if, whenever the open set $\mathfrak{G}$ contains $\mathfrak{A}_0$, there exists an open set $\mathfrak{G}_0$, $\mathfrak{A}_0 \subset \mathfrak{G}_0 \subset \mathfrak{G}$, such that $\mathfrak{A}\cdot\mathfrak{G}_0 \neq 0$ implies $\mathfrak{A} \subset \mathfrak{G}$. A map is said to be covering, irredundant, continuous, according as the determining class α covers $\mathfrak{S}$, is densely distributed in $\mathfrak{S}$, is continuous, respectively. By removing from $\mathfrak{S}$ and all its subsets those points which do not belong to $\mathfrak{S}_\alpha$, we may convert any map $m(\mathfrak{R},\mathfrak{S},\alpha)$ into a covering map $m(\mathfrak{R},\mathfrak{S}_\alpha,\alpha)$. By removing from $\mathfrak{S}$ and all its subsets the points of an appropriate open set $\mathfrak{G}$, obtained by an obvious transfinite construction, we may convert any map $m(\mathfrak{R},\mathfrak{S},\alpha)$ into an irredundant map $m(\mathfrak{R},\mathfrak{S}^*,\alpha^*)$, where $\mathfrak{S}^*$ is the complement of $\mathfrak{G}$ and α^* is the class of all sets $\mathfrak{A}\cdot\mathfrak{S}^*$. The existence of a map $m(\mathfrak{R},\mathfrak{S},\alpha)$ implies the existence of a single-valued function $f(s)$ defined for all points s in $\mathfrak{S}_\alpha$ with $\mathfrak{R}$ as its range, the set of points s which are the antecedents of a fixed point r in $\mathfrak{R}$ being a set $\mathfrak{A}$ in α. Since the connection between maps and functional correspondences has been extensively investigated,[13] we shall omit further remarks on the subject here. In view of the fact that a map $m(\mathfrak{R},\mathfrak{S},\alpha)$ may assign more than one antecedent in $\mathfrak{S}$ to a given point in $\mathfrak{R}$, it is convenient to define the index of a map as the least cardinal number $\mathfrak{c}$ such that $\mathfrak{c} + 1$ is exceeded[14] by the cardinal number of no set $\mathfrak{A}$ in α.

A space $\mathfrak{Q}$ is said to be an extension of the space $\mathfrak{R}$ if $\mathfrak{R}$ is equivalent to a subset $\mathfrak{Q}(\mathfrak{R})$ of $\mathfrak{Q}$. In case $\mathfrak{Q}(\mathfrak{R})$ is a proper subset of $\mathfrak{Q}$, the space $\mathfrak{Q}$ is said to be a proper extension of $\mathfrak{R}$; and in case $\mathfrak{Q}(\mathfrak{R})$ is dense in $\mathfrak{Q}$, the space $\mathfrak{Q}$ is said to be an immediate extension of $\mathfrak{R}$. A space $\mathfrak{R}$ is said to be absolutely closed if there exists no proper immediate extension of $\mathfrak{R}$. If $\mathfrak{Q}$ is an extension of $\mathfrak{R}$, the existence of a map $m(\mathfrak{Q},\mathfrak{S},\beta)$ implies the existence of a map $m(\mathfrak{R},\mathfrak{S},\alpha)$ where α is a subclass of β; and, if α is a subclass of β, the existence of a map $m(\mathfrak{R},\mathfrak{S},\alpha)$ implies the existence of an extension $\mathfrak{Q}$ of $\mathfrak{R}$ and the existence of a map $m(\mathfrak{Q},\mathfrak{S},\beta)$. In order that $\mathfrak{Q}$ be a proper extension of $\mathfrak{R}$, it is necessary and sufficient that α be a proper subclass of β; and, in order that $\mathfrak{Q}$ be an immediate extension of $\mathfrak{R}$, it is necessary and sufficient that every open subset of $\mathfrak{S}$ which contains a set in β contain also a set in α. We see further that when $\mathfrak{Q}$ is an immediate extension of $\mathfrak{R}$

the map $m(\mathfrak{Q},\mathfrak{S},\beta)$ is irredundant if and only if the map $m(\mathfrak{R},\mathfrak{S},\alpha)$ is irredundant.

If the theory of maps is to be useful as a general tool in the study of arbitrary spaces, it is obviously necessary for us to discover some universal but specialized type of mapping space. We shall show that the Boolean spaces can be so employed as mapping spaces.

THEOREM V_1. *If $\mathfrak{B}$ is a Boolean space, α a class of mutually disjoint closed sets densely distributed in $\mathfrak{B}$, and $\mathfrak{R}$ the space $\mathfrak{R}(\mathfrak{B},\alpha)$, then there exists an irredundant map $m(\mathfrak{R},\mathfrak{B},\alpha)$.*

THEOREM V_2. *If $\mathfrak{R}$ is an arbitrary space, there exists an irredundant map $m(\mathfrak{R},\mathfrak{B}^*,\alpha^*)$ where $\mathfrak{B}^*$ is a Boolean space. This map is constructed in the following manner: if v is an arbitrary determining system of neighborhoods in $\mathfrak{R}$, we first construct the Boolean algebra $A[v]$ consisting of all the sets obtained by applying to finite classes of neighborhoods in v and their complements finite sequences of the operations of forming the union and intersection, observing that the class $\mathfrak{a}$ of all sets in $A[v]$ which are nowhere dense in $\mathfrak{R}$ is an ideal in $A[v]$; we next construct the spaces $\mathfrak{B} = \mathfrak{S}(A[v])$ and $\mathfrak{B}^* = \mathfrak{S}(A[v]/\mathfrak{a})$ in accordance with Theorem IV_1, obtaining $\mathfrak{B}^*$ from $\mathfrak{B}$ by the removal of an open set $\mathfrak{G}(\mathfrak{a})$ in accordance with Theorem IV_3; we then define α as the class of all sets $\mathfrak{A}$ where $\mathfrak{A} = \mathfrak{A}(r)$ is the intersection of all the open-closed sets $\mathfrak{E}(U)$ in $\mathfrak{B}$ which correspond to neighborhoods U in v containing the arbitrary point r in $\mathfrak{R}$; and we define α^* as the class of all sets $\mathfrak{A}^* = \mathfrak{A}^*(r) = \mathfrak{A}(r)\cdot\mathfrak{B}^*$; finally, the map $m(\mathfrak{R},\mathfrak{B}^*,\alpha^*)$ is given by the appropriate correspondence between the points r in $\mathfrak{R}$ and the sets $\mathfrak{A}^*(r)$ in α^*.*

THEOREM V_3. *If $m(\mathfrak{R},\mathfrak{B}^*,\alpha^*)$ is an irredundant map of an arbitrary space $\mathfrak{R}$ on a Boolean space $\mathfrak{B}^*$ determined by the class α^*, then there exists a determining system v of neighborhoods in $\mathfrak{R}$ such that the given map is equivalent to the one constructed from v by the process described in Theorem V_2.*

From these theorems we see that the theory of arbitrary spaces $\mathfrak{R}$ is coextensive with the theory of densely distributed classes α of mutually disjoint closed sets in Boolean spaces $\mathfrak{B}$; but the latter theory, by Theorems IV_1–IV_3, is coextensive with the theory of Boolean algebras. In other words, there is complete mathematical equivalence between the theory of Hausdorff topology and the theory of Boolean algebras.

We shall now point out a number of special theorems which follow more or less directly from the preceding discussion.

THEOREM VI. *Every immediate extension $\mathfrak{Q}$ of a space $\mathfrak{R}$ can be constructed by selecting an irredundant map $m(\mathfrak{R},\mathfrak{B},\alpha)$ of $\mathfrak{R}$ in a Boolean space $\mathfrak{B}$ and choosing a class β in $\mathfrak{B}$ of which α is a subclass, the spaces $\mathfrak{Q},\mathfrak{B}$, and the class β being then related by a map $m(\mathfrak{Q},\mathfrak{B},\beta)$.*

THEOREM VII. *A space $\mathfrak{R}$ is absolutely closed if and only if a fixed, but entirely arbitrary, irredundant map $m(\mathfrak{R},\mathfrak{B},\alpha)$ of $\mathfrak{R}$ in a Boolean space $\mathfrak{B}$ is a covering map.*

THEOREM VIII. *A space $\mathfrak{R}$ is regular if and only if a fixed, but entirely arbitrary, irredundant map $m(\mathfrak{R},\mathfrak{B},\alpha)$ of $\mathfrak{R}$ in a Boolean space $\mathfrak{B}$ is a continuous map.*

THEOREM IX. *A space $\mathfrak{R}$ is bicompact if and only if a fixed, but entirely arbitrary, irredundant map $m(\mathfrak{R},\mathfrak{B},\alpha)$ of $\mathfrak{R}$ in a Boolean space $\mathfrak{B}$ is a covering continuous map; in other words, if and only if $\mathfrak{R}$ is absolutely closed and regular.*[15]

THEOREM X. *An arbitrary space $\mathfrak{R}$ has at least one absolutely closed immediate extension $\mathfrak{Q}$.*[16]

THEOREM XI. *A space $\mathfrak{R}$ is bicompact if and only if every closed subset of $\mathfrak{R}$ is absolutely closed.*[17]

THEOREM XII. *If $\mathfrak{R}$ is an arbitrary space, the cardinal number $d(\mathfrak{R})$ which is the least among the indices of all (irredundant) maps $m(\mathfrak{R},\mathfrak{B},\alpha)$, where $\mathfrak{B}$ is a Boolean space, is a topological invariant with the following properties:*

(1) *$d(\mathfrak{R}) = 0$ if and only if $\mathfrak{R}$ is equivalent to a subset of a Boolean space;*

(2) *$d(\mathfrak{R})$ is equal to the Urysohn-Menger dimension of $\mathfrak{R}$ if one of these cardinal numbers is finite and $\mathfrak{R}$ is a compact metric space (bicompact space of character not exceeding $\aleph_0$).*

The space $\mathfrak{R}$ has at least one absolutely closed immediate extension $\mathfrak{Q}$ such that $d(\mathfrak{Q}) = d(\mathfrak{R})$.[18]

The further consideration of the invariant $d(\mathfrak{R})$ will obviously involve an elaborate discussion along the lines of the existing dimension theory. I propose to return to the question at a later time.

[1] The first part of the theory was presented to the American Mathematical Society, February 25, 1933 (*Bull. Amer. Math. Soc.*, abstract No. 39–3–86, received January 24, 1933) and before mathematical gatherings at Brown, Johns Hopkins, Yale and Harvard Universities. The second part of the theory was presented before a meeting of the National Academy of Arts and Sciences in Cambridge, Mass., November 22, 1933.

[2] See for instance the postulates proposed by Huntington, *Trans. Am. Math. Soc.*, **5**, 288–309 (1904), or those proposed by del Re, *Algebra della Logica* (Naples, 1907), 8, and discussed by B. A. Bernstein, *University of California Publications in Mathematics*, **1**, 87–89 (1914). The writer has obtained a new set of postulates with several features of interest and has also considered generalized Boolean algebras in which no unit or universal element is postulated.

[3] See for instance B. L. van der Waerden, *Moderne Algebra*, **1** (Leipzig, 1930), Chapter 3.

[4] The symmetric difference of two elements, b and c, is the unique element x such that $x + bc = b + c$, $x(bc) = 0$. In terms of negation, it may be written $x = bc' + b'c$. The symmetric difference, more familiar to writers on the theory of sets than to writers on Boolean algebras, has many interesting abstract properties which seem to have passed unnoticed.

[5] In the case where A is an algebra of propositions and $\mathfrak{a}$ is a prime ideal in A, the homomorphism from A to $A/\mathfrak{a}$ may be interpreted as assigning to each element in A a truth-value compatible with the logic of material implication. In the case where A is an algebra of abstract sets and $\mathfrak{a}$ is a prime ideal in A, the homomorphism from A to $A/\mathfrak{a}$ may

be interpreted as assigning to each element in A a numerical measure which has the value 0 or 1 and which is additive in the restricted sense. For the construction of such a measure, see A. Tarski, *Fundamenta Mathematicae*, **15**, 42–50 (1930).

[6] This theorem was announced by the writer in explicit terms in the abstract cited under footnote one. A more general theorem was subsequently obtained and published by Garrett Birkhoff, *Proc. Camb. Phil. Soc.*, **29**, 441–464 (1933), Theorem 25.2, without knowledge of this abstract. It should be noted that when the algebra A contains just one element, 0, the set $\mathfrak{E}$ is void and coincides with $\mathfrak{E}(0)$.

[7] See Hausdorff, *Grundzüge der Mengenlehre* (first edition, Leipzig, 1914); Alexandroff and Urysohn, *Mathemat. Annal.*, **92**, 258–266 (1924); Alexandroff, *Mathemat. Annal.*, **92**, 267–274 (1924); Alexandroff and Urysohn, *Mémoire sur les espaces topologiques compacts*, Verhandelingen der Koninklijke Akademie van Wetenschappen te Amsterdam, Eerste Sectie, Deel XIV, No. 1 (1929), cited below as *A U*.

[8] A space is said to be totally disconnected if each pair of distinct points determines a representation of the space as the union of two disjoint closed sets each of which contains one point of that pair.

[9] We shall use the word "equivalent" to signify "topologically equivalent." In case A contains just one element, the space $\mathfrak{S}(A)$ is void.

[10] See Tychonoff, *Mathemat. Annal.*, **102**, 544–561 (1930), especially Theorems I and II, cited below as *T*.

[11] Many of the propositions which follow can be extended to spaces more general than Hausdorff spaces; but the extension involves complications which would only obscure the primary facts in a condensed presentation such as this.

[12] See Alexandroff, *Mathemat. Annal.*, **96**, 555–571 (1926), cited below as *A*.

[13] See *A*.

[14] In a more detailed exposition, one should consider the distinction between the cases where the bound $\mathfrak{c} + 1$ is attained and those where it is not.

[15] This theorem is not new. It is proved very simply, in the second form stated here, in *A U*, 43–48, 49–50.

[16] This theorem was proposed in *A U*, p. 52, but was not proved. With the word "immediate" deleted, the theorem was established in *T*, Theorem III; but, as our Theorem XI shows, the insertion of this word yields a theorem stronger than that given in *T*.

[17] The necessity of the stated condition is quite trivial; its sufficiency was proved in *A U*, p. 50, for spaces of character not exceeding $\aleph_0$ but not for any other case.

[18] There exists a map $m(\mathfrak{R},\mathfrak{B},\alpha)$ of index $d(\mathfrak{R})$; but such maps can be further classified according to the suggestion made above.[15] The result in (2) is due in its complete form to Kuratowski, *Fundamenta Mathematicae*, **18**, 285–292 (1932).

GROUPS WHOSE SQUARES CONSTITUTE CYCLIC SUBGROUPS

BY G. A. MILLER

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS

Communicated January 22, 1934

It is known[1] that no dicyclic group can be generated by the squares of the operators of a group and that a necessary and sufficient condition that a dihedral group is composed of the operators which are squares under another group is that the order of its cyclic subgroup of index 2 is not divisible by 4 and that every odd prime number which divides this order is congruent to unity modulo 4. If a cyclic group H of odd order is composed of the squares of the operators of a group G whose order exceeds the order of this cyclic group but which is not the direct product of a group having this cyclic group as the group composed of the squares of its operators and the abelian group of order 2^m and of type $(1,1,1,\ldots)$ then G is non-abelian. The quotient group G/H is abelian and of type $(1,1,1,\ldots)$ since the order of each of its operators divides 2. Therefore each of the operators of G which is not in H transforms the operators of at least one of the Sylow subgroups of H into their inverses and hence the order of G cannot exceed $h.2^n$, h being the order of H and n being the number of the distinct prime numbers which divide h.

For the sake of brevity it will be assumed in what follows that the groups which are said to have the property that the squares of their operators constitute a given subgroup are not the direct products of such groups and of an abelian group of order 2^m and of type $(1,1,1\ldots)$, since all such direct products have this property in common. The main results of the preceding paragraph can therefore be stated as follows: *The number of the groups which have the property that the squares of their operators constitute a given cyclic subgroup of odd order h is equal to the number of the subgroups of the abelian group of order 2^n and of type $(1,1,1,\ldots)$, n being the number of the distinct prime numbers which divide h.* In particular, the number of such groups of order $2h$ is 2^n-1.

According to this theorem there is a finite number of groups which separately satisfy the condition that the squares of their operators constitute a given cyclic group of odd order. On the contrary, there is always an infinite number of distinct groups which separately satisfy the condition that the squares of their operators constitute a given cyclic subgroup of an even order. In both cases all the possible groups are either direct products of groups in which the order of the given cyclic subgroup is a power of a prime number or they are obtained by establishing an isomorphism between such groups. Hence our problem is now practically reduced to the determination of the possible groups when the order of the given cyclic sub-

group is of the form 2^m. Therefore we shall assume in what follows that the order of H is of this form.

The group G must therefore contain an invariant cyclic group of order 2^{m+1} and all the operators of H are either invariant under G or they are transformed into their inverses thereunder, since the operators of this invariant cyclic group of order 2^{m+1} cannot be transformed in more than four different ways under G because they must be transformed either according to an operator of order 2 or according to the identity in its group of isomorphisms. Hence there results the following theorem: *If the squares of the operators of a group constitute the cyclic group of order 2^m then the operators of this cyclic subgroup are either invariant under the entire group or they are transformed into their inverses thereunder.* If the order of G exceeds 2^{m+1} then G is non-abelian and every co-set of G with respect to a cyclic subgroup of order 2^{m+1} involves operators whose orders divide 4. In what follows it will be assumed that these conditions are satisfied.

We shall first determine the possible groups under which every operator of H is invariant. Two cases present themselves. In one of these G contains a cyclic group of order 2^{m+1} whose operators are invariant under it while in the other it contains no such cyclic subgroup. In both cases every co-set of G with respect to such an invariant cyclic subgroup involves operators of order 2 whenever the order of H exceeds 2, as will be assumed in what follows since the case when the order of H is 2 has been fully treated recently.[2] None of these operators of order 2 except the one which appears in H is invariant under G since it is assumed that G is not the direct product of a group such that the square of its operators constitute H and of the abelian group of order 2^m and of type $(1,1,1,\ldots)$. As a non-invariant operator of G has only two conjugates under G there results the following theorem: *There is one and only one group of order $2^{m+\alpha}, \alpha > 0$, which has the property that the squares of its operators constitute the cyclic group of order 2^m and that every operator of this cyclic subgroup is invariant under the group.*

A necessary and sufficient condition that this group contains invariant operators of order 2^{m+1} is that α is odd. The group in question can always be constructed as follows: Start with the cyclic group of order 2^{m+1}. If α is odd extend this cyclic group by an operator of order 2 which is commutative with each of its operators and then extend the group thus obtained by an operator of order 2 which is commutative with every operator of the given cyclic group but transforms the given operator of order 2 into itself multiplied by the operator of this order in the given cyclic group. If $\alpha > 3$ and odd this process may be repeated by starting with the group thus obtained instead of with the given cyclic group. When α is even the first extending operator transforms all the operators of the given cyclic subgroup of order 2^{m+1} into their $2^m + 1$ powers but otherwise the process is the same as before. It remains to consider the case when the operators

of H are transformed into their inverses under G. Every group which comes under this case involves one of the groups just considered as a subgroup of index 2.

Each of the operators which must be added to this subgroup is of order 2 or of order 4. When all of these added operators are of the same order the given subgroup of index 2 is abelian and hence of order 2^{m+1}. It will be proved that there are two and only two such groups. They are the dihedral and the dicyclic group of order 2^{m+2}. In each of the remaining groups the operators which are added to the given subgroup of index 2 have two different orders, for if they were of the same order they would have a common square and hence would transform every operator of this subgroup into its inverse and therefore this subgroup would be abelian. Each of the $2^{\alpha-1}$ cyclic subgroups of order 2^{m+1} contained in G is invariant under G and its operators are transformed either into their inverses or into their $2^m - 1$ by the added operators. Hence a set of generating operators of highest order in the given subgroup of index 2 under G cannot be transformed in more than 2^α different ways by the added operators.

As the group of inner isomorphisms of G is also of order 2^α when α is even it remains only to show in this case that an operator of order 2 which is added could not transform the operators of the given subgroup of index 2 in the same manner as such an operator of order 4. This results from the fact that this subgroup contains an odd number of pairs of operators of order 2 such that each pair appears in a co-set with respect to a cyclic subgroup of order 2^{m+1}, and that if an added operator of order 2 is commutative with each of the operators of such a pair the operators of order 4 resulting from the same co-set do not have this property, and vice versa. It therefore results that when α is even there are two groups of order $2^{m+\alpha+1}$ which involve the same subgroup of index 2 generated by its operators of order 2^{m+1}. For the same reason we conclude that when α is odd there are two groups under which the operators of order 2^{m+1} which are invariant under the given subgroup of index 2 are transformed into their inverses, while there is only one such group under which these operators are transformed into their $2^m - 1$ powers. Hence there results the following theorem: *When $\alpha > 0$ is even there are two and only two groups of order $2^{m+\alpha+1}$, $m > 1$, which have the property that the squares of their operators constitute the cyclic subgroup of order 2^m and are transformed into their inverses under it. When α is odd there are three such groups.* This theorem does not apply to the groups which are the direct products of groups such that the squares of their operators constitute the cyclic group of order 2^m and of an abelian group of type (1,1,1,...).

By means of the preceding theorems it is possible to determine the number of the groups of a given order which have the property that the squares of their operators constitute a given cyclic subgroup of order 2^m, $m > 1$,

including those which are the direct products of such a group and an abelian group of type (1,1,1, . . .). There are exactly α such groups of order $2^{m+\alpha}$ under each of which the operators of this cyclic subgroup are invariant. The number of these groups under which the operators of this cyclic subgroup are transformed into their inverses is $5^{\alpha}/2-2$ when α is even and $5(\alpha-1)/2$ when α is odd. It should be noted that this number is independent of the value of $m > 1$. When the order of H is of the form $2^m k$, where k is odd, the number of such groups may be found by determining the possible direct products and isomorphisms between the different groups represented by these expressions and the groups having the cyclic group of order k as the group of their squares.

To exhibit the usefulness of these theorems in the determination of all the groups of a given order it may be noted that there is one such group of order 64 which has only one square, there are seven which have two squares, there are twelve such groups which have the cyclic group of order 4 as the subgroup of their squares, there are eight which have the cyclic group of order 8 as the subgroup of their squares, while five such groups have the cyclic group of order 16, and one has the cyclic group of order 32, as such a subgroup. Hence 34 of the groups of order 64 have some cyclic group, including the identity, as the subgroup composed of their squares. Eleven of the 14 groups of order 16 have the same property. In one of the remaining groups the group of the squares constitute the four group, while the squares do not constitute a subgroup in each of the two remaining ones.

[1] G. A. Miller, these PROCEEDINGS, **19**, 1054 (1933).
[2] G. A. Miller, *Amer. Jour. Math.*, **55**, 417 (1933).

AVERAGE DISTRIBUTION OF ARBITRARY MASSES UNDER GROUP TRANSLATIONS

BY S. BOCHNER

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY

Communicated January 30, 1934

The object of the present note is to give a partial extension of the recent theorems of G. D. Birkhoff and J. v. Neumann concerning time averages of functions under stationary transformations.[1] We shall consider not point functions but general additive set functions, which may represent masses concentrated even in arbitrary null sets; but we shall have to restrict the transformations to translations in groups. In the special case of a mass concentrated in a single point our result reduces to a theorem of H. Weyl[2] on equidistribution of numbers mod 1.

We denote by G a space, by $x,y, \ldots$ its points, by $A, B, \ldots$ its subsets, by μA a Lebesgue measure in G for which $\mu G = 1$, by $d\omega$ the space element; by $f(x)$ a point function and by $F(A)$ a set function; by $T_t x$ a group, $T_{t_1+t_2} = T_{t_1} T_{t_2}$, of one-to-one measure preserving transformations of G into itself, which is either continuous ($-\infty < t < \infty$), or discontinuous ($t = 0, \pm 1, \pm 2, \ldots$); by $f_t(x), f_{t,s}(x)$ the expressions

$$\frac{1}{t}\int_0^t (T_t x)dt, \frac{1}{t-s}\int_0^t f(T_t x)dt \text{ or } \frac{1}{t}\sum_{\nu=1}^{t} f(T_\nu x), \frac{1}{t-s}\sum_{\nu=1}^{t} f(T_\nu x), \quad (1)$$

and by $F_t(A)$, $F_{t,s}(A)$ corresponding time averages of $F(A)$.

Birkhoff's theorem says that, for any $f(x) \subset L_1$, the function $f_t(x)$ has a limit for almost all x as $t \longrightarrow \infty$. Denoting the limit by $f^*(x)$, the argument of the theorem also shows that

$$\int |f_t(x) - f^*(x)|^p d\omega \longrightarrow 0 \text{ as } t \longrightarrow \infty, \text{ if } f(x) \subset L_p, p \geqq 1. \quad (2)$$

We shall say that we are in the ergodic case if $f^*(x)$ is a constant for any $f(x) \subset L_1$.

The theorem (2), although not a direct consequence of Birkhoff's theorem, is considerably weaker. But it is of intrinsic importance, as independent investigations of Koopmann, v. Neumann and others have revealed. v. Neumann has proved the following. If G is metric separable and complete, and if all Borel sets are measurable under μA, then we also have

$$\int |f_{t,s}(x) - f^*(x)|^2 dx \longrightarrow 0 \text{ as } t - s \longrightarrow \infty \text{ if } f(x) \subset L_2; \quad (3)$$

moreover, the ergodic case is characterized by the fact that every function $f(x) \subset L_2$ which for each t fulfils the relation $f(T_t x) = f(x)$ for almost all x is a constant.

If we introduce, for measurable A, the set function

$$F(A) = \int_A f(x)d\omega, \quad (4)$$

then we deduce from (2) and (3) the relations

$$F_{t,s}(A) \longrightarrow F^*(A) \text{ as } t - s \longrightarrow \infty \quad (5)$$

$$\int |dF_t(A) - dF^*(A)| \longrightarrow 0 \text{ as } t \longrightarrow \infty; \quad (6)$$

and the ergodic case is characterized by

$$F^*(A) = F(G)\cdot\mu A. \quad (7)$$

Now let $F(A)$ denote any bounded additive set function, defined on a Borel field of sets A which is invariant under the transformations T_t. A Borel field is thus defined that it contains G and the meet and join of any two of its sets. Under fairly general conditions, such a function $F(A)$ is

the difference of two non-negative functions of the same kind. We shall therefore assume throughout that $F(A) \geqq 0$, in which case $F(A)$ may be interpreted as a mass distribution. If $F(A)$ is completely continuous it is the (unique) sum of two parts $F'(A)$, $F''(A)$, one of which is "regular" and other "singular." The regular part $F'(A)$ is the integral (4) of a function $f(x) \subset L_1$, and thus stands for a mass distribution with assignable density. Whereas the singular part $F''(A)$ is the very opposite; there exists a null set A_0, on which the whole mass is concentrated, namely, $F''(A_0A) = F''(A)$ for any A, and in particular: $F''(A_0) = F''(A)$. Our aim is to find conditions, not affecting the generality of the singular part of $F(A)$, under which the relations (5), (7) remain valid.

We have dropped the relation (6) entirely, because it fails to hold in the simplest cases. For instance, consider for the real numbers mod 1 the translations $T_tx = x + t\gamma$, where γ is irrational. In the discontinuous case (t runs over integers), a mass 1 concentrated in a single point distributes equally over $0 \leqq x < 1$, but it is easily seen that

$$\int |dF_t(A) - dF_s(A)| = 1 \qquad \text{for } t > 0,\ s > 0.$$

In a two-dimensional torus the same is true even in the continuous case, if the translations form an irrational angle.

But we shall show that the relations (5), (7) do hold, at least, if the transformations T_t are translations in a group.

We assume that G is a compact metric separable topological group, and that the measure μA is the group-invariant Haar measure which, owing to the compactness, may be shown to be the same with respect to translations from the left and from the right; furthermore we assume that the transformations T_t are all translations. The decisive (and, as we shall later see, indispensable) consequence of the latter assumption is this: that any property of smoothness of the function $f(x)$ holds for the functions (1) uniformly in t and s. For instance, if $f(x)$ is continuous, then the functions (1) are equally uniformly continuous, and if $f(x) \subset L_p$, then the functions (1) are a semicompact set of functions $\subset L_p$. In particular, if $g(x)$ is continuous, it follows from (3), by an application of the diagonal argument, that

$$g_{t,s}(x) \longrightarrow g^*(x), \text{ as } t - s \longrightarrow \infty, \text{ uniformly in } x \subset G. \qquad (8)$$

Now let $F(A)$ be a (non-negative) set function and let every set of its Borel field be subdividable into a finite number of sets of the field with arbitrarily small diameters. As usual, we can define the Riemann Stieltjes integral $\int g(x)\, dF(A)$ for all continuous functions $g(x)$. From (8) we conclude

$$\int g(x)\, dF_{t,s}(A) = \int g_{-s,-t}(x)\, dF(A) \longrightarrow \int g^*(x) dF(A). \qquad (9)$$

We consider a set A which is "squarable": to any $\epsilon > 0$ there exist continuous functions $g(x)$, $h(x)$ such that

$$\begin{aligned} &g(x) \leqq 1 \text{ for } x \subset A,\ g(x) \leqq 0 \text{ for } x \not\subset A \\ &h(x) \geqq 1 \text{ for } x \subset A,\ h(x) \geqq 0 \text{ for } x \not\subset A \end{aligned} \tag{10}$$

$$\int (h(x) - g(x))d\omega \leqq \epsilon. \tag{11}$$

From (9) and (10) we follow

$$\int g^*(x)dF(A) \leqq \underline{\lim}\ F_{t,s}(A) \leqq \int h^*(x)\ dF(A).$$

The desired relation (5) would follow if we could infer from (11) that $\int (h^*(x) - g^*(x))dF(A)$ is small for small ϵ. We cannot infer this in the non-ergodic case, but in the ergodic case we have $h^*(x) - g^*(x) = \int (h(x) - g(x))d\omega \leqq \epsilon$. Hence we conclude:

If T_t are ergodic translations in a compact separable metric group then for any bounded non-negative additive set function $F(A)$ the relations (5) *and* (7) *hold on any set A which is squarable with respect to the invariant measure of the group.*

It is easy to construct an example showing that the theorem fails if A is not squarable but only measurable.

In the non-ergodic case we shall apply another argument. Let G be the two dimensional torus represented by pairs of real numbers $x = (u,v)$, each reduced mod. 1, and considered as a group under vector translations. And let $F(A)$ be a bounded non-negative additive interval function defined primarily for the intervals in the square $0 \leqq u < 1$, $0 \leqq v < 1$. The Fourier coefficients

$$c_{pq} = \int e^{-2\pi i(pu+qv)}dF(A) \quad p,q = 0, \pm 1, \pm 2, \ldots$$

have the following important property.[3] If for a sequence $F_k(A)$, $k = 1,2, \ldots$ the corresponding coefficients c^k_{pq} are convergent as $k \longrightarrow \infty$ (no uniformity with respect to the subscripts is required), then the functions $F_k(A)$ are essentially convergent to a function $F^*(A)$ of the same type. Precisely, there exists an interval function $F^*(A)$, such that $F_k(A)$ converges to $F^*(A)$ on each interval A, for which $F(A)$ is continuous; and these are practically all intervals. Moreover, the functions $F_k(A)$ and $F^*(A)$ may be extended to other Borel sets and the convergence still holds for squarable sets A for which $F^*(A)$ is continuous.

If starting from an interval function $F(A)$ we form the interval functions $F_{t,s}(A)$, then, by an easy computation, for fixed p,q, the (p,q)-th Fourier coefficient of $F_{t,s}(A)$ is $c_{pq} \cdot e_{t,s}(0)$ where $e(x) = e^{2\pi i(pu+qv)}$. Applying (8), we find that the Fourier coefficients of $F_{ts}(A)$ converge as $t - s \longrightarrow \infty$. Hence we have, for interval functions $F(A)$ on the torus, the following theorem.

In the non-ergotic case, there exists a limit function $F^*(A)$ such that the relation (5) *holds on every squarable set A for which $F^*(A)$ is continuous.*

The work of F. Peter and H. Weyl[4] enables us to generalize our argument from the torus to any compact separable metric group. Let $e(x)$ denote one of the character functions forming a complete orthogonal system on the group. It may be shown that also for set functions on G the set of Fourier coefficients

$$c = \int e(t^{-1})dF(A)$$

determines the function $F(A)$ in the same way as on the torus, that the convergence of the Fourier coefficients involves the essential convergence of the set functions and that the Fourier coefficients of $F_{ts}(A)$ are convergent as $t - s \longrightarrow \infty$; and this convergence again follows from the relation (8) if applied to the functions $e(t)$. The details of this Fourier analysis on groups will be supplied in another context.

Finally we shall give an example of a discontinuous group of transformations T_t on the two dimensional (u,v)-torus, which form the ergodic case but are not translations and for which the relation (7) fails to hold. We consider a unimodular transformation $T = T_1$, of the (u,v) plane into itself, and we consider the discontinuous group generated by T. It is not a translation group but meets all other requirements. The origin is a fixed point. A mass in this point is therefore fixed and cannot become equidistributed. But there are unimodular transformations which are ergodic; these are all such transformations whose proper values are not roots of unity. The following proof is due to v. Neumann. A function $f(x) \subset L_2$ on the torus is determined by its Fourier series

$$\sum a(p,q)e^{2\pi i(pu+qv)} \qquad \sum |a(p,q)|^2 < \infty.$$

The Fourier coefficients of $f(T^k x)$ are a permutation of the coefficients $a(p,q)$ which arises if the transformation T^{-k} is applied to the integers (p,q). And now it can be easily seen that there is no non-constant common solution of the relations $f(T^k x) = f(x)$, unless some proper value of the transformation T is a root of unity.

[1] See the list of papers quoted in *Bull. Amer. Math. Soc.*, **39**, 852–853 (1933); and Khintchine, *A., Math. Annal.*, **107**, 485–488 (1933).

[2] *Math. Annalen*, **77**, 313 (1916).

[3] In the case of Fourier integrals instead of Fourier series the details are to be found in Bochner, S., *Math. Annalen*, **108**, 378–410 (1933).

[4] *Ibid.*, **97**, 737–755 (1927).

ENTIRE FUNCTIONS AND BOREL'S DIRECTIONS

BY G. VALIRON

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY

Communicated December 30, 1933

1. We consider a Dirichlet series

$$F(z) = \sum_1^\infty a_n + e^{\lambda_n z},\ \lambda_{n+1} > \lambda_n,\ \lambda_1 \geqq 0,\ z = x + iy, \tag{1}$$

where

$$\overline{\lim_{n \to \infty}} \frac{\log n}{\lambda_n} = D < \infty.$$

We put $|a_n| = A_n$ and denote by $x_a(F)$ the abscissa of absolute convergence; we know that

$$-D \leqq x_a(F) + \overline{\lim_{n \to \infty}} \frac{\log A_n}{\lambda_n} \leqq 0.$$

We denote by $\mu(x, F)$ the maximum of $A_n e^{\lambda_n x}$, $(n = 1, 2, \ldots)$ and by $M(x, F)$ the maximum of $|F(x + iy)|$, $-\infty < y < +\infty$, $x = \text{const.} < x_a(F)$. We have

$$\mu(x, F) \leqq M(x, F) \leqq K + K'\mu(x + D + \epsilon),\ \epsilon > 0, \tag{2}$$

K and K' being constants which depend on $F(z)$ and ϵ. If $x_a(F) = +\infty$, since $\log \mu(x, F)$ is an indefinitely increasing convex function of x, we obtain

$$M(x, F) < \mu(x + D + \epsilon),\ \epsilon > 0,\ x > x(\epsilon). \tag{3}$$

The order of growth of $M(x, F)$ is then connected with that of $\mu(x, F)$. This order of $M(x, F)$ may be called the *linear* order of the entire function $F(z)$, the ordinary order of $F(z)$ being a *circular* order which is less than or equal to the linear one. We consider here only the case where

$$\tau = \overline{\lim_{x \to \infty}} \frac{\log \log M(x, F)}{x} > 0$$

which was first studied by Ritt.[1] Mandelbrojt and Gergen[2] have investigated the properties of the Julia's lines of $F(z)$ which are parallel to the real axis, in order to extend certain theorems due to Pólya.[3] I will show that we may approach this problem by the method which I employed in the case of Taylor's series.[4] Denote by $V(t)$ a real continuous function of the real variable t, defined for $t \geqq 1$ and such that

$$\overline{\lim_{t \to \infty}} \frac{\log M(t, F)}{V(t)} = 1 \quad , \qquad V(t) \geqq \log M(t, F),$$

$$V(t) > t^2 \text{ if } t > T_0 \geqq 1,$$

$$\int_{D+1}^{\infty} \frac{W(t - D)}{W(t)}\, dt < \infty, \qquad \text{where } W(t) = e^{V(t)}.$$

As an example, we may choose $V(t)$ in the following manner:
(i) If $\tau < \infty$, $V(t)$ is for each t the greatest of the two numbers $e^{(\tau - \epsilon)t}$, $\epsilon > 0$, $\tau - \epsilon > 0$ and $\log M(t, F)$;

(ii) If $\tau = \infty$, we take

$$\frac{\log V(t)}{t} = \text{maximum of } \frac{\log^+ \log^+ M(u, F)}{u} \text{ for } u \leqq t.$$

In all cases, we introduce the function

$$\Omega(u) = \int_1^{\infty} \frac{e^{ut}}{W(t)}\, dt \tag{4}$$

which is an increasing function of u. If $\nu(u)$ denotes the maximum of $\frac{e^{ut}}{W(t)}$, we see that, provided u is sufficiently large, we have

$$\frac{\nu(u)}{2u} < \Omega(u) < 2u\nu(u). \tag{5}$$

The inequalities (2), (3), (5) show us that the Dirichlet's series

$$f(z) = \sum_1^n a_n \Omega(\lambda_n) e^{\lambda_n z} \tag{6}$$

which is associated with $F(z)$ admits a finite abscissa of absolute convergence $x_a(f)$ such that $-D \leqq x_a(f) \leqq D$. And if $x < x_a(f)$ we may write

$$f(z) = \int_1^{\infty} \frac{F(t + z)}{W(t)}\, dt. \tag{7}$$

Therefore, if

$$|F(x_0 + iy_0 + x + iy)| < W(x - D)$$

in the half strip $|y| \leqq \epsilon$, $x \geqq X_0$, $f(z)$ may be continued along the half straight $y = y_0$, $x < x_0$. Hence:

I. *If we consider the analytic continuation of $f(z)$ along the parallels to the real axis, and if $z_0 = x_0 + iy_0$ is a singularity of the function thus defined, there exists a sequence of points $z_n = x_n + iy_n$ $(n = 1, 2, \ldots)$ such that when $n \longrightarrow \infty$*

$$y_n \longrightarrow y_0, \quad x_n \longrightarrow \infty$$

and

$$|F(z_n)| > W(x_n - D - x_0 - o(1)).$$

From this proposition we deduce the existence of a Borel's line. To obtain it in the case where $\tau < \infty$ we transform the half strip $|y - y_0| \leqq \frac{\pi}{2\alpha}$, $\alpha < \tau$, $x \geqq 0$ into a sector of magnitude π within which the order of the transformed function is greater than 1; this sector contains a Borel's direction (5) and, by returning to the strip, we obtain the following result:

II. *If z_0 is a singularity of the function $f(z)$ defined as above, there exists a sequence of circles Γ_n, $|z - z_n| \leqq o|(1)$ where $z_n = x_n + iy$, $x_n \longrightarrow \infty$, $y' = const.$, $|y' - y_0| \leqq \frac{\pi}{2\tau}$ such that within Γ_n the function $F(z)$ assumes*

$$e^{x_n(\tau + \epsilon_n)},\ |\epsilon_n| = o(1),\ \epsilon_n = \epsilon_n(Z) \tag{8}$$

times every value Z for which

$$\log|Z| < e^{x_n(\tau - o(1))},\ \log|Z - Z_n| > -e^{x_n(\tau - o(1))} \tag{9}$$

the points Z_n tending to a limit when $n \longrightarrow \infty$.

The same transformation gives similar results when $\tau = \infty$ but y' is then equal to y_0 and in (8) and (9), $\tau + \epsilon_n$, $\tau - o(1)$ are replaced by $\tau_n \longrightarrow \infty$. But in this case it is more convenient to apply the following generalization of Schottky's theorem:

III. *Suppose that $f(z)$ is holomorphic in the circle $|z| < 1$, that a and b satisfy the conditions $|a| < A$, $|b| < A$, $|a - b| > \frac{1}{A}$ and that $f(z) - a$, $f(z) - b$ have fewer than $P + 1$ roots in the circle. Suppose also that for c real, $0 \leqq c < 1$*

$$\frac{1}{c}\int_0^c \log^+|f(x)|\,dx < B,\ 0 \leqq x \leqq c.$$

Then, we have

$$\log|f(z)| < \alpha_k P \log\frac{2}{c} + \beta_k \log A + \gamma_k B + \delta_k$$

for $|z| \leqq k < 1$, α_k, β_k, γ_k, δ_k being numbers which depend only on k.

We obtain thus:

IV. *If $\tau = \infty$, if z_0 is a singularity of $f(z)$ defined as above, there are circles Γ_n, $|z - z_n| < \frac{hx_n}{\log V(x_n - l)}$, $z_n = x_n + iy_n$, $x_n \longrightarrow \infty$, $y_n \longrightarrow y_0$, $l > D + x_0$, $h > 0$, such that within Γ_n, $F(z)$ assumes at least $V(x_n - l)$ times all values Z for which $|Z| < V(x_n - l)$, $|Z - Z_n| > 1/V(x_n - l)$ where $V(x)$ is the function of the example* (ii).

We can sum up by saying that $y = y_0$ is a Borel's line of the order of $V(x)$, for the number of zeros of $F(z) - Z$ in Γ_n has also an upper bound of the order of $V(x)$. If the properties of the sequence $\{\lambda_{n+1} - \lambda_n\}$ or other considerations enable us to conclude that each straight $y = y_0$ contains a singularity of $f(z)$, we deduce in this case of the infinite order τ that each such straight line is a Borel's line of $F(z)$.

We can also start from a function $f(z)$ with a singular set cutting the plane and obtain a family of associated entire functions admitting each line $y = y_0$ as a Borel's line.

These properties correspond to summation by the Borel-Laplace integral; we may also generalize certain properties connected with the summation by Mittag-Leffler's integral used first for Dirichlet series by M. Riesz.[6]

2. The proposition III leads immediately to the following theorem:

V. *Let us consider an entire function $f(z)$ of positive finite (circular) order ρ, let $\rho(r)$ be a proximate order L of $f(z)$ $[\rho(r) \longrightarrow \rho,\ \rho'(r)r \log r \longrightarrow 0]$ and let us suppose that*

$$J(\varphi) = \int_1^\infty \frac{\log^+ |f(re^{i\varphi})|}{r^{1+\rho(r)}}\, dr,\ z = re^{i\varphi},$$

is divergent for some values φ, so that the values φ for which $J(\varphi) = \infty$ constitute intervals of lengths at least equal to $\frac{\pi}{\rho}$. Then, the extremities of such intervals (if they exist) give Borel's directions of order $\rho(r)$ divergent.[7]

It may be observed that the hypothesis of theorem III concerning the modulus of $f(z)$ may be replaced by others in which we give a bound of $f(z)$ at points whose mutual distances satisfy certain conditions; but my method does not give the best result.

3. The use of the Hankel and Mittag-Leffler integral

$$\int_C \frac{V(t)dt}{z-t}$$

where $V(t)$ is an entire function of the kind that I called "fonctions orientées"[8] and C is a contour constituted by an arc of circle $|t| = \text{const.}$, and by two receding curves lying in two different regions where $V(t)$ tends to zero exponentially, permits to construct easily irregular entire functions analogous to Mittag-Leffler's functions. We may thus obtain a sequence of entire functions $f(z)$ with the same proximate order L, $\rho(r)$, such that the indicated function of order

$$H(\varphi) = \overline{\lim_{r \to \infty}} \frac{\log^+ |f(re^{i\varphi})|}{r^{\rho(r)}} \qquad (10)$$

is equal to $\cos \rho\varphi$ if $|\varphi| \leqq \frac{\pi}{2\rho}$ and to 0 for other values of φ, but the r

for which the quotient in the right-hand side of (10) is near to $H(\varphi)$ are different for the different functions. Then, applying the method that I used in the case $\rho(r) = \rho + \frac{d}{\log r}$ [9] we see that:

VI. *Given a subtrigonometric function of order $\rho' H(\varphi)$, which is always positive or zero, and a proximate order $L\ \rho(r)$ tending to ρ, there exists an entire function $f(z)$ for which we have the equality* (10).

[1] "On Certain Points in the Theory of Dirichlet Series," *Amer. Jour. Math.*, **50**, 73 (1928).

[2] "On Entire Functions Defined by a Dirichlet Series," *Ibid.*, **53**, 1 (1931).

[3] "Untersuchungen über Lücken und Singularitäten von Potenzreihen," *Math. Zeitschr.*, **29**, 549 (1929).

[4] "Méthodes de sommation et directions de Borel," *Annali R. Scuola nor. di Pisa*, **2**, 2, (1933).

[5] "Recherches sur le Théorème de M. Borel," *Acta math.*, **52**, 67 (1928).

[6] "Sur la représentation analytique des fonctions définies par des séries de Dirichlet," *Ibid.*, **35**, 253 (1911).

[7] See Valiron, "Sur les directions de Borel des fonctions méromorphes d'ordre fini," *Jour. math.*, **9**, 10, 457 (1931), and a note by Rauch, *Bull. Soc. math*, **61** (1933).

[8] "Sur les fonctions entières d'ordre fini, . . . ," *Annales fac. sc. Toulouse*, **3**, 5, 117 (1913).

[9] "Fonctions entières et fonctions convexes," *Bull. soc. math.*, **60**, 117 (1933). For the definition of subtrigonometric functions, see my cited paper of Journal math.

THE REDUCTION OF DEGENERATE QUADRATIC DIFFERENTIAL FORMS

BY TRACY YERKES THOMAS

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY

Communicated February 2, 1934

1. Consider a quadratic differential form

$$\sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} g_{\alpha\beta} dx^{\alpha} dx^{\beta} \tag{1}$$

in a domain T such that

$$|x^1 - a^1| < b^1, \ldots, |x^n - a^n| < b^n,$$

where the a's are constants and the b's are positive constants. Assume that the coefficients $g_{\alpha\beta}$ are analytic functions of the variables x^{α} in T.

Let the rank of the matrix $||g_{\alpha\beta}||$ be r such that $1 \leqq r < n - 1$. Since $||g_{\alpha\beta}||$ is symmetric it follows from a theorem in Algebra that this matrix

contains a diagonal determinant D of order r which does not vanish identically in T. We can therefore assume that D is not equal to zero at any point of T by a suitable restriction if necessary of this domain. Without loss of generality D can be taken to occupy the position in the upper left-hand corner of the matrix $||g_{\alpha\beta}||$ since a mere relettering of the variables will in any case accomplish this result. Such a relettering of the variables is equivalent to a coördinate transformation; in fact if the non-vanishing diagonal determinant D of order r corresponds to the rows and columns $\alpha_1, \ldots, \alpha_r$ we can by the coördinate transformation $x^{\alpha_i} = \bar{x}^{\beta_i}$, $x^\alpha = \bar{x}^\beta$ where $\beta_1, \ldots, \beta_r = 1, \ldots, r$ bring this determinant into the upper left-hand corner of the matrix.

In this paper we investigate the existence of analytic transformations of the coördinates x^α such that (1) is reduced to the form

$$\sum_{a=1}^{r} \sum_{b=1}^{r} h_{ab} dy^a dy^b, \tag{2}$$

where the coefficients h_{ab} are analytic functions of the variables $y^1, \ldots, y^n$ in a domain

$$|y^1 - a^1| < b_*^1, \ldots, |y^n - a^n| < b_*^n \tag{3}$$

contained in T. Such transformations will reduce effectively the degenerate quadratic differential form (1) to a non-degenerate form in r variables $y^1, \ldots, y^r$ and $n - r$ parameters $y^{r+1}, \ldots, y^n$.

2. It is required to find a solution of the proper type of the system

$$\sum_{\beta=1}^{n} g_{\alpha\beta}(x) \frac{\partial x^\beta}{\partial y^i} = 0, \quad \begin{pmatrix} \alpha = 1, \ldots, n \\ i = r+1, \ldots, n \end{pmatrix}. \tag{4}$$

Now since the matrix $||g_{\alpha\beta}||$ is of rank r and furthermore since the determinant of order r in the upper left-hand corner of this matrix does not vanish in the domain T by hypothesis, there must exist identical relations of the form

$$g_{\mu b} = \sum_{a=1}^{r} g_{ab} A_\mu^a \qquad \begin{pmatrix} b = 1, \ldots, r \\ \mu = r+1, \ldots, n \end{pmatrix}, \tag{5}$$

$$g_{\mu\nu} = \sum_{a=1}^{r} g_{a\nu} A_\mu^a = \sum_{a,b=1}^{r} g_{ab} A_\mu^a A_\nu^b, \ (\mu, \nu = r+1 \ldots, r). \tag{6}$$

Eliminate the quantities $g_{\mu b}$ and $g_{\mu\nu}$ in the left members of (5) and (6) from the system (4). We obtain

$$\sum_{b=1}^{r} g_{ab} \frac{\partial x^b}{\partial y^\nu} + \sum_{b=1}^{r} \sum_{\mu=r+1}^{n} g_{ab} A_\mu^b \frac{\partial x^\mu}{\partial y^\nu} = 0, \quad \begin{pmatrix} a = 1, \ldots, r \\ \nu = r+1, \ldots, n \end{pmatrix}, \tag{7}$$

$$\sum_{a,b=1}^{r} g_{ab} A_\mu^a \frac{\partial x^b}{\partial y^\nu} + \sum_{a,b=1}^{r} \sum_{\sigma=r+1}^{n} g_{ab} A_\mu^a A_\sigma^b \frac{\partial x^\sigma}{\partial y^\nu} = 0, \ (\mu,\nu = r+1, \ldots, r). \tag{8}$$

Since the determinant $|g_{ab}|$ is not equal to zero in T, the equations (7) can be solved in this domain so as to yield

$$\frac{\partial x^a}{\partial y^\nu} = -\sum_{\mu=r+1}^{n} A_\mu^a \frac{\partial x^\mu}{\partial y^\nu}, \quad \begin{pmatrix} a = 1, \ldots, r \\ \nu = r+1, \ldots, n \end{pmatrix}. \tag{9}$$

When the left members of (9) are substituted into (8) these latter equations are satisfied identically. Hence the problem of obtaining a solution of the system (4) is reduced to that of obtaining a solution of (9). The quantities A_μ^a in (9) are analytic functions of the variables x^α in T; this follows from the corresponding hypothesis on the coefficients $g_{\alpha\beta}$ and the fact that the A_μ^a are uniquely determined by the equations (5).

3. Take first the case $r = n - 1$. Putting $x^n = y^n$ in (9) these equations become

$$\frac{\partial x^k}{\partial y^n} = -A_n^k(x^1, \ldots, x^{n-1}, y^n), \ (k = 1, \ldots, n-1) \tag{10}$$

and can in fact be regarded as a system of ordinary differential equations. In accordance with the existence theorem for such systems the equations (10) admit a solution

$$x^k = \varphi^k(y^1, \ldots, y^{n-1}, y^n), \ (k = 1, \ldots, n-1), \tag{11}$$

such that $x^k = y^k$ for $y^n = a^n$ where the φ's are analytic in a domain (3) and can be solved uniquely for the $n - 1$ variables $y^1, \ldots, y^{n-1}$ in this domain. Hence the equations (11) taken in conjunction with the equation $x^n = y^n$ define an analytic coördinate transformation of the domain (3) of T such that (1) is reducible to the form (2).

THEOREM. *If the rank of the matrix* $||g_{ab}||$ *is* $n - 1$ *there always exists an analytic coördinate transformation of the* n *variables* x^α *which reduces the degenerate form* (1) *to the non-degenerate form* (2).

As a matter of fact there exists a one parameter family of such transformations since the functions φ^k can be considered to involve analytically a parameter y_0^n such that $|y_0^n - a^n| < b_*^n$, the quantity y_0^n being associated with the initial value of the variable y^n.

4. Assume $r = n - 2$. Then by a slight formal modification of the equations of §3 we can show that (1) can be transformed into (2) with $r = n - 1$. Now, however, the determinant $|h_{ab}|$ vanishes identically in the domain (3). But the determinant of order $n - 2$ in the upper left-hand corner of the matrix $||h_{ab}||$ does not vanish in (3); this follows directly from the type of coördinate transformation at which we arrive as a result of the process of §3 and the fact that the determinant of order $n - 2$ in the upper left-hand corner of the original matrix $||g_{\alpha\beta}||$ is assumed to be different from zero in T.

Now repeat the process of §3 with reference to the above degenerate

form (2) for $r = n - 1$ when this form is regarded as a quadratic differential form in $n - 1$ variables $y^1, \ldots, y^{n-1}$ depending on a parameter y^n. We thus arrive at the form (2) for $r = n - 2$ with coefficients h_{ab} which are analytic in a domain of the type (3) and with a determinant $|h_{ab}|$ not equal to zero in this domain. As this result can be extended to cases $r < n - 2$ we have the following:

THEOREM. *If $r < n - 1$ it is possible to reduce* (1) *to the non-degenerate form* (2) *by a succession of coördinate transformations: First a coördinate transformation in the n variables x^α resulting in a degenerate form in $n - 1$ variables, then a transformation of the $n - 1$ variables of this latter form by which it is reduced to a form in $n - 2$ variables, ..., and finally a transformation in $r + 1$ variables by which the non-degenerate form* (2) *is obtained.*

5. Now assume $r < n - 1$ and consider the question of replacing the succession of transformations in the theorem of the last section by a single coördinate transformation in the n variables x^α which will reduce (1) to the non-degenerate form (2). This is not always possible since integrability conditions arise! Differentiating (9) with respect to y^μ, interchanging the indices μ,ν and subtracting in the usual manner, we obtain these conditions in the form

$$\sum_{\alpha,\beta=r+1}^{n} \left\{ \sum_{b=1}^{r} \left[\frac{\partial A^a_\alpha}{\partial x^b} A^b_\beta - \frac{\partial A^a_\beta}{\partial x^b} A^b_\alpha \right] - \frac{\partial A^a_\alpha}{\partial x^\beta} + \frac{\partial A^a_\beta}{\partial x^\alpha} \right\} \frac{\partial x^\alpha}{\partial y^\mu} \frac{\partial x^\beta}{\partial y^\nu} = 0, \quad (12)$$

where $a, b = 1, \ldots, r$ and $\mu, \nu = r + 1, \ldots, n$.

Suppose now that at a point P of the domain T the determinant $|\partial x^\alpha / \partial y^\mu|$ of order $n - r$ is equal to zero. Introduce the notation $B^i_\mu = -A^i_\mu$ when $i = 1, \ldots, r$; $\mu = r + 1, \ldots, n$ and $B^i_\mu = \delta^i_\mu$ when $i,\mu = r + 1, \ldots, n$. Then we have

$$\frac{\partial x^i}{\partial y^\nu} = \sum_{\mu=r+1}^{n} B^i_\mu \frac{\partial x^\mu}{\partial y^\nu}, \qquad \begin{pmatrix} i = 1, \ldots, n \\ \nu = r + 1, \ldots, n \end{pmatrix}$$

from (9), or in matrix form

$$\left\| \frac{\partial x^i}{\partial y^\nu} \right\| = \left\| B^i_\mu \right\| \cdot \left\| \frac{\partial x^\mu}{\partial y^\nu} \right\|, \qquad \begin{pmatrix} i = 1, \ldots, n \\ \mu,\nu = r + 1, \ldots, n \end{pmatrix}. \quad (13)$$

Since by the above assumption the rank of the $n - r$ rowed square matrix $\|\partial x^\mu / \partial y^\nu\|$ is less than $n - r$ at the point P and since the rank R of the matrix $\|\partial x^i / \partial y^\nu\|$ in the left member of (13) cannot exceed the rank of the matrix $\|\partial x^\mu / \partial y^\nu\|$ it follows that $R < n - r$ at P. Hence the determinant $|\partial x^i / \partial y^j|$ where $i,j = 1, \ldots, n$ must be equal to zero at the point P with the result that the solution of (9) cannot, in the neighborhood of P, represent an analytic transformation with unique inverse, i.e., a coördinate transformation. Hence we must assume that the above determinant $|\partial x^\mu / \partial y^\nu|$ is not equal to zero at some point P of the domain T and we can

without real loss of generality assume that $|\partial x^\mu/\partial y^\nu|$ does not equal zero at the point with coördinates $x^\alpha = a^\alpha$ used in the definition of T. Hence the integrability conditions (12) become

$$\sum_{b=1}^{r}\left[\frac{\partial A^a_\alpha}{\partial x^b}A^b_\beta - \frac{\partial A^a_\beta}{\partial x^b}A^b_\alpha\right] - \frac{\partial A^a_\alpha}{\partial x^\beta} + \frac{\partial A^a_\beta}{\partial x^\alpha} = 0, \begin{pmatrix} a = 1, \ldots, r \\ \alpha,\beta = r+1, \ldots, n \end{pmatrix}. \quad (14)$$

Now put

$$x^{r+1} = y^{r+1}, \ldots, x^n = y^n \quad (15)$$

in (9) so that these equations become

$$\frac{\partial x^k}{\partial y^\nu} = -A^k_\nu(x^1, \ldots, x^r, y^{r+1}, \ldots, y^n), \quad \begin{pmatrix} k = 1, \ldots, r \\ \nu = r+1, \ldots, n \end{pmatrix} \quad (16)$$

Assuming that (14) is satisfied in T, the system (16) is completely integrable and hence admits a solution

$$x^k = \varphi^k(y^1, \ldots, y^r, y^{r+1}, \ldots, y^n), (k = 1, \ldots, n), \quad (17)$$

such that $x^k = y^k$ for $y^{r+1} = a^{r+1}, \ldots, y^n = a^n$. Also the φ^k in (17) are analytic functions in a domain (3) and can be solved uniquely for the r variables $y^1, \ldots, y^r$ in this domain. Hence (15) and (17) define an analytic coördinate transformation of the domain (3) in consequence of which the form (1) is reduced to the non-degenerate form (2).

THEOREM. *When the rank of the matrix* $\|g_{\alpha\beta}\|$ *is less than* $n - 1$ *there exists an analytic coördinate transformation of the* n *variables* x^α *which reduces the degenerate form* (1) *to the non-degenerate form* (2) *if, and only if, the conditions* (14) *are satisfied.*

Corresponding to the equations (11) the above equations (17) belong to an $n - r$ parameter family, the parameters being $y_0^{r+1}, \ldots, y_0^n$ such that

$$|y_0^{r+1} - a^{r+1}| < b_*^{r+1}, \ldots, |y_0^n - a^n| < b_*^n.$$

Since the A^a_μ are uniquely determined by (5) when the matrix $\|g_{\alpha\beta}\|$ is of rank r, the conditions (14) are immediately expressible as conditions on the coefficients of the form (1).

ERRATUM

In the contribution by W. E. Castle, these PROCEEDINGS, **20,** February, 1934, on p. 102, line 15 from the bottom, for "any eggs" read "many eggs."